Integration Techniques

The definite integral

$$\int_a^b f(x)$$

Is the **area under the graph** of the function $f(x) \geq 0$, over the x-axis on $[a, b]$.

To compute the definite integral $\longrightarrow$ divide the interval into n subintervals and approximate the function by a constant.

Then, compute the area of n rectangles. If $f \simeq f(x_i)$ at the i-th interval, with all the intervals with same length, Δx, then

$$\int_a^b f(x) \simeq f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Diff.: Given $F(x)$, find $f(x)$ $\leftrightarrow$ **Integration:** Given $f(x)$, find $F(x)$

$$\frac{dF(x)}{dx} = f(x).$$

A function $F(x)$ solving the second problem is called an **antiderivative primitive** or an **indefinite integral** of $f(x)$.

1. $\int_a^b c_1 f + c_2 g =$
   $c_1 \int_a^b f + c_2 \int_a^b g$
2. $\int_a^b f = \int_a^c f + \int_c^b f$
3. $\int_a^b f = -\int_b^a f$
4. $\int_a^a f = 0$
5. $\int_a^b fg \neq \int_a^b f \int_a^b g$
6. $f \geq g \Rightarrow \int_a^b f \geq \int_a^b g$
7. $f \geq 0 \Rightarrow \int_a^b f \geq 0$
   if $f \leq 0 \Rightarrow \int_a^b f \leq 0$
8. $\left| \int_a^b f \right| \leq \int_a^b |f|$
9. $m \leq f(x) \leq M, \ \forall x \in [a, b] \Rightarrow$
   $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

Properties of the integral

First Mean Value Theorem for integrals

Let f be continuous on $[a, b]$, then $\exists\, x_0 \in [a, b]$ such that

$$\int_a^b f = f(x_0)(b-a).$$

$\frac{1}{b-a}\int_a^b f \to$ average of f over $[a, b]$.

Second Mean Value Theorem for integrals

Let f be continuous on $[a, b]$ and g integrable such that g does not change sign on $[a, b]$, then $\exists\, x_0 \in [a, b]$ such that

$$\int_a^b fg = f(x_0)\int_a^b g.$$

Basic antiderivatives

$$\int x^n = \frac{x^{n+1}}{n+1} + c, \;\; n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int e^{ax} = \frac{1}{a} e^{ax} + c$$

$$\int \sin x = -\cos x + c$$

$$\int \cos x = \sin x + c$$

$$\int \frac{1}{\cos^2 x} = \tan x + c$$

$$\int \frac{1}{\sin^2 x} = -\cot x + c$$

$$\int \frac{1}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sinh x = \cosh x + c$$

$$\int \cosh x = \sinh x + c$$

Integration Techniques

Integration by change of variables (CV)

- Definite integral: $\displaystyle\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt$
- Indefinite integral: $\displaystyle\int f(x)dx = \int f(g(t))g'(t)dt$

$\longrightarrow$ undo the change

Integration by parts (IBP): $\displaystyle\int udv = uv - \int vdu$

- Definite integral: $\displaystyle\int_a^b fg' = fg\Big|_a^b - \int_a^b f'g$
- Indefinite integral: $\displaystyle\int fg' = fg - \int f'g$

Rational functions: partial fraction decomposition

$$\int \frac{P(x)}{Q(x)} dx \;\rightarrow\; P,\; Q \text{ polynomials}$$

- If $deg(P) \geq deg(Q) \rightarrow$ **divide the polynomials:** $P(x) = Q(x)C(x) + R(x) \rightarrow$

$$\int \frac{P(x)}{Q(x)} dx = \int C(x) + \int \frac{R(x)}{Q(x)} dx.$$

- $\int \frac{R(x)}{Q(x)} dx$ with $deg(R(x)) < deg(Q(x))$:

  I) First, check that the integral is not immediate:

  ln type $\rightarrow \int \frac{2x+3}{x^2+3x+8} dx = \ln|x^2+3x+8| + c.$

  arctan type $\rightarrow \int \frac{dx}{x^2+8} = \frac{1}{\sqrt{8}} \arctan \frac{x}{\sqrt{8}} + c.$

  II) If not $\rightarrow$ **Do partial fraction decomposition.**

Rational functions: partial fraction decomposition

Factor in denominator	Term in partial fraction decomposition
$x-b$	$\dfrac{A}{x-b}$
$(x-b)^k$	$\dfrac{A_1}{x-b}+\dfrac{A_2}{(x-b)^2}+\cdots\dfrac{A_k}{(x-b)^k},\quad k=1,2,3,\cdots$
$(x-a)^2+b^2$	$\dfrac{Ax+B}{(x-a)^2+b^2}$
$\left((x-a)^2+b^2\right)^k$	$\dfrac{A_1x+B_1}{(x-a)^2+b^2}+\cdots+\dfrac{A_kx+B_k}{\left((x-a)^2+b^2\right)^k},\quad k=1,2,3,\cdots$

For each factor in the denominator add the corresponding term of the table and compute the unknowns $(A,\ B, A_1,\ B_1,\ A_2,\ B_2,\cdots)$ by setting equal denominators. After, compute the integrals of each term.

Irrational functions or integrals involving roots

Do a change of variables that eliminates the roots.

$$\int R\left[\left(\frac{ax+b}{cx+d}\right)^{p_1/q_1}, \cdots, \left(\frac{ax+b}{cx+d}\right)^{p_r/q_r}\right] \rightarrow$$

$$t^m = \frac{ax+b}{cx+d}, \; m = lcm(q_1, \cdots, q_r).$$

$R = \dfrac{P}{Q}$ is a rational function of its variables, $P, \; Q$ are polynomials.

$lcm \rightarrow$ least common multiple.

Integrals involving trigonometric functions

- $\int \sin^{2n} x,\ \int \cos^{2n} x \longrightarrow$
  double angle formulas: $\cos 2x = \cos^2 x - \sin^2 x$
- $\int \sin^{2n+1} x = \int \sin^{2n} x \sin x = \int (1-\cos^2 x)^n \sin x$
- $\int \cos^{2n+1} x = \int \cos^{2n} x \cos x = \int (1-\sin^2 x)^n \cos x$
- $\int \sin mx \cos nx \longrightarrow$ trig formulas
- $\displaystyle\int R(\sin x, \cos x) \longrightarrow$
  R odd in $\sin x \longrightarrow t = \cos x$
  R odd in $\cos x \longrightarrow t = \sin x$
  R even in $\cos x$ and $\sin x \longrightarrow t = \tan x$
  Rest of problems $\longrightarrow t = \tan x/2$,
  $$\left(\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2}{1+t^2}dt\right)$$

Some change of variables

1. $\int R(x, \sqrt{x^2 + a^2}) \to x = a \tan t$
2. $\int R(x, \sqrt{x^2 - a^2}) \to x = \frac{a}{\cos t}$
3. $\int R(x, \sqrt{a^2 - x^2}) \to x = a \sin t$