UNIVERSIDAD CARLOS III DE MADRID

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Set prepared by
Venancio Alvarez
Arturo de Pablo
Domingo Pestana
José Manuel Rodríguez
Eva Tourís

English version by
Marina Delgado

Contents

1 INTEGRATION IN ONE VARIABLE

Problem 1.1 Find the following antiderivatives:

1. $\int x \operatorname{tg}^2(2x)\,dx$
2. $\int \operatorname{tg}^3 x \sec^4 x\,dx$
3. $\int \frac{\sqrt{x}+1}{x+3}\,dx$
4. $\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$
5. $\int \frac{x^2}{(x-1)^3}\,dx$
6. $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$
7. $\int \frac{\sin^2 x \cos^5 x}{\operatorname{tg}^3 x}\,dx$
8. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$
9. $\int \mathrm{e}^x \sin \pi x\,dx$
10. $\int \frac{dx}{\cos^4 x}$
11. $\int \sin^2 x\,dx$
12. $\int \sin^4 x\,dx$
13. $\int \cos^2 x\,dx$
14. $\int \cos^6 x\,dx$
15. $\int \sin^2 x \cos^2 x\,dx$
16. $\int \frac{dx}{3+\sqrt{2x+5}}$
17. $\int \sqrt{\frac{x-1}{x+1}}\,dx$
18. $\int \operatorname{arctg} \sqrt[3]{x}\,dx$
19. $\int \sqrt{\sqrt{x}+1}\,dx$
20. $\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$
21. $\int \sqrt{2+\mathrm{e}^x}\,dx$
22. $\int \mathrm{e}^{\sin x} \cos^3 x\,dx$
23. $\int \sin^5 x\,dx$
24. $\int \cos^3 x \sin^2 x\,dx$
25. $\int \operatorname{tg}^2 x\,dx$
26. $\int \operatorname{tg}^3 x\,dx$
27. $\int x^3 \sqrt{1-x^2}\,dx$
28. $\int \frac{\sin x + 3\cos x}{\sin x \cos x + 2\sin x}\,dx$
29. $\int \frac{\sin x + 3\cos x}{\sin x + 2\cos x}\,dx$
30. $\int \operatorname{tg}^2(3x)\sec^3(3x)\,dx$
31. $\int \frac{4x^4 - x^3 - 46x^2 - 20x + 153}{x^3 - 2x^2 - 9x + 18}\,dx$
32. $\int \cos(\log x)\,dx$
33. $\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x}+\mathrm{e}^x+2}\,dx$
34. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$
35. $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$
36. $\int \frac{2}{x^2-2x+2}\,dx$
37. $\int \frac{dx}{\cos^2 x}$
38. $\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$
39. $\int \frac{x}{(x^2+1)^{5/2}}\,dx$
40. $\int x^2(1-x^2)^{-3/2}\,dx$
41. $\int \sqrt{\mathrm{e}^x - 1}\,dx$
42. $\int \frac{2x^2+3}{x^2(x-1)}\,dx$

43. $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$ 44. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$ 45. $\displaystyle\int x^2\sqrt{x-1}\,dx$

46. $\displaystyle\int \sec^6 x\,dx$ 47. $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$ 48. $\displaystyle\int \frac{dx}{\mathrm{e}^x-4\mathrm{e}^{-x}}$

49. $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$ 50. $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$ 51. $\displaystyle\int \mathrm{e}^x\cos 2x\,dx$

52. $\displaystyle\int x^2\log x\,dx$ 53. $\displaystyle\int \sin^3 x\cos^2 x\,dx$ 54. $\displaystyle\int \cos^4 x\,dx$

55. $\displaystyle\int \mathrm{tg}^4 x\,dx$ 56. $\displaystyle\int \sec^3 x\,dx$ 57. $\displaystyle\int \frac{dx}{1-\sin x}$

58. $\displaystyle\int \sin(\log x)\,dx$ 59. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$ 60. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

61. $\displaystyle\int \frac{dx}{\sqrt{\mathrm{e}^{2x}-1}}$ 62. $\displaystyle\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x}+2\mathrm{e}^x+2}\,dx$ 63. $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$

64. $\displaystyle\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}}$ 65. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$ 66. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$

67. $\displaystyle\int x^m\log x\,dx$ 68. $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$ 69. $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$

70. $\displaystyle\int \cos^2(\log x)\,dx$ 71. $\displaystyle\int (\log x)^3\,dx$ 72. $\displaystyle\int x(\log x)^2\,dx.$

Hint: IBP means integration by parts and CV change of variables.

1. IBP $dv=\mathrm{tg}^2(2x)dx=(\sec^2(2x)-1)dx$.
2. CV $t=\mathrm{tg}\ x$.
3. CV $t=\sqrt{x}$.
4. CV $t=\sqrt{1-(x+1)^2}$.
5. Do partial fraction decomposition or expand x^2 in powers of $x-1$.
6. CV $x=\sec t$.
7. As the integrand is odd in sine, CV $t=\cos x$.
8. The derivative of the denominator almost appears in the numerator.
9. IBP twice using $dv=\mathrm{e}^x dx$.
10. As the integrand is even in sine and cosine, CV $t=\mathrm{tg}\,x$.
11, 12, 13, 14 and 15. Use double angle formulas.
16. CV $t=3+\sqrt{2x+5}$ or $t=\sqrt{2x+5}$.
17. CV $t=\sqrt{(x-1)/(x+1)}$.
18. CV $x=t^3$, after do IBP with $u=\mathrm{arctg}\,t$.
19. CV $t=\sqrt{\sqrt{x}+1}$.
20. CV $t=\sqrt{x+2}$.
21. CV $t=\sqrt{\mathrm{e}^x+2}$.
22. CV $t=\sin x$, after do IBP twice with $dv=\mathrm{e}^t dt$.
23. As the integrand is odd in sine, CV $t=\cos x$.
24. As the integrand is odd in cosine, CV $t=\sin x$.
25. $\mathrm{tg}^2 x=\sec^2 x-1$, or apply the CV $t=\mathrm{tg}\,x$.
26. CV $t=\mathrm{tg}\,x$.
27. CV $t=\sqrt{1-x^2}$.
28. CV $t=\mathrm{tg}(x/2)$.
29. CV $t=\mathrm{tg}\,x$.
30. CV $t=\sin(3x)$.
31. The denominator is $(x-2)(x-3)(x+3)$.

32. IBP twice using $dv = dx$ or use the CV $t = \log x$.
33. CV $t = \mathrm{e}^x$.
34. CV $t = \sqrt{1 + x^{1/3}}$.
35. CV $x = \operatorname{tg} t$.
36. The denominator is equal to $(x-1)^2 + 1$.
37. It is immediate.
38. CV $x + 2 = t^3$.
39. It is immediate. Can be integrated also using the CV $t = x^2 + 1$.
40. CV $x = \sin t$.
41. CV $t = \sqrt{\mathrm{e}^x - 1}$.
42. Decompose in partial fractions.
43. It is immediate. Can be integrated also using the CV $t = \sqrt{1 - \sqrt{x}}$.
44. Multiply and divide by $1 - \cos x$.
45. IBP twice taking the derivative of the polynomial or use the CV $t = \sqrt{x-1}$.
46. CV $t = \operatorname{tg} x$.
47. CV $t = 1 + x^2$.
48. CV $t = \mathrm{e}^x$.
49. CV $t^2 = 1 + x$.
50. CV $t^3 = 1 - x$.
51. IBP twice using $dv = \mathrm{e}^x dx$.
52. IBP $u = \log x$.
53. CV $t = \cos x$.
54. Use double angle formulas.
55. CV $t = \operatorname{tg} x$.
56. CV $t = \sin x$.
57. Multiply and divide by $1 + \sin x$.
58. IBP twice using $dv = dx$ or use the CV $t = \log x$.
59. CV $x = \sin t$.
60. It is immediate (you can also use the CV $t^2 = 1 + x^2$).
61. CV $t^2 = \mathrm{e}^{2x} - 1$.
62. CV $t = \mathrm{e}^x$.
63. The denominator is equal to $(x-1)^2(x+1)^2$.
64. CV $t^6 = 1 - 2x$.
65. CV $x = 3\sin t$.
66. $x^2 + x + 1 = (x + 1/2)^2 + 3/4$.
67. IBP $u = \log x$.
68. CV $t = \sin x$.
69. CV $t^2 = x^3$.
70. Use double angle formulas. Next do IBP twice using $dv = dx$ or use the CV $t = 2\log x$.
71. IBP $u = (\log x)^3$.
72. IBP $u = (\log x)^2$.

Solution:
1. $\frac{1}{2}\, x \operatorname{tg}(2x) + \frac{1}{4}\, \log|\cos(2x)| - \frac{1}{2}\, x^2 + c$.
2. $\frac{1}{6}\, \operatorname{tg}^6 x + \frac{1}{4}\, \operatorname{tg}^4 x + c$.
3. $2\sqrt{x} + \log|x+3| - 2\sqrt{3}\, \operatorname{arctg} \sqrt{\frac{x}{3}}\, + c$.
4. $\frac{1}{3}\, (1 - (x+1)^2)^{3/2} - (1 - (x+1)^2)^{1/2} + c$.
5. $\frac{-1}{2(x-1)^2} - \frac{2}{x-1} + \log|x-1| + c$.
6. $\frac{1}{2}\, x\sqrt{x^2-1} + \frac{3}{2}\, \log|x + \sqrt{x^2-1}\,| + c$.
7. $\frac{1}{7}\, \cos^7 x + \frac{1}{5}\, \cos^5 x + \frac{1}{3}\, \cos^3 x + \cos x + \frac{1}{2}\, \log(1 - \cos x) - \frac{1}{2}\, \log(1 + \cos x) + c$.
8. $-\log|\sin x + \cos x| + c$.
9. $\frac{1}{1+\pi^2}\, \mathrm{e}^x(\sin \pi x - \pi \cos \pi x) + c$.
10. $\frac{1}{3}\, \operatorname{tg}^3 x + \operatorname{tg} x + c$.
11. $\frac{1}{2}\, x - \frac{1}{4}\, \sin 2x + c$.
12. $\frac{3}{8}\, x - \frac{1}{4}\, \sin 2x + \frac{1}{32}\, \sin 4x + c$.
13. $\frac{1}{2}\, x + \frac{1}{4}\, \sin 2x + c$.
14. $\frac{5}{16}\, x + \frac{1}{4}\, \sin 2x + \frac{3}{64}\, \sin 4x - \frac{1}{48}\, \sin^3 2x + c$.
15. $\frac{1}{8}\, x - \frac{1}{32}\, \sin 4x + c$.
16. $\sqrt{2x+5}\, - 3\, \log\left(3 + \sqrt{2x+5}\,\right) + c$.
17. $\frac{-1}{t-1} + \log|t-1| - \frac{1}{t+1} - \log|t+1| + c$, where $t = \sqrt{(x-1)/(x+1)}$.
18. $x \operatorname{arctg}(x^{1/3}) - \frac{1}{2}\, x^{2/3} + \frac{1}{2}\, \log(x^{2/3} + 1) + c$.
19. $\frac{4}{5}\, (\sqrt{x} + 1)^{5/2} - \frac{4}{3}\, (\sqrt{x} + 1)^{3/2} + c$.
20. $x - 2\sqrt{x+2} + 2\log(\sqrt{x+2} + 1) + c$.
21. $2\sqrt{2 + \mathrm{e}^x} + \sqrt{2}\, \log(\sqrt{2 + \mathrm{e}^x} - \sqrt{2}\,) - \sqrt{2}\, \log(\sqrt{2 + \mathrm{e}^x} + \sqrt{2}\,) + c$.

22. $-(1-\sin x)^2\,\mathrm{e}^{\sin x}+c.$
23. $\frac{-1}{5}\,\cos^5 x+\frac{2}{3}\,\cos^3 x-\cos x+c.$
24. $\frac{1}{3}\,\sin^3 x-\frac{1}{5}\,\sin^5 x+c.$
25. $\operatorname{tg} x-x+c.$
26. $\frac{1}{2}\,\operatorname{tg}^2 x+\log|\cos x|+c.$
27. $\frac{1}{5}\,(1-x^2)^{5/2}-\frac{1}{3}\,(1-x^2)^{3/2}+c.$
28. $\log\left|\operatorname{tg}\frac{x}{2}\right|-2\log\left(\operatorname{tg}^2\frac{x}{2}+3\right)+\frac{2}{\sqrt{3}}\,\operatorname{arctg}\frac{\operatorname{tg}(x/2)}{\sqrt{3}}+c.$
29. $\frac{1}{5}\,\log|\operatorname{tg} x+2|-\frac{1}{10}\,\log\left(\operatorname{tg}^2 x+1\right)+\frac{7}{5}\,x+c.$
30. $\frac{1}{12}\,\sec^3 3x\operatorname{tg} 3x-\frac{1}{24}\,\sec 3x\operatorname{tg} 3x-\frac{1}{24}\,\log|\sec 3x+\operatorname{tg} 3x|+c.$
31. $2x^2+7x+3\log|x-2|-4\log|x-3|+5\log|x+3|+c.$
32. $\frac{1}{2}\,x\cos(\log x)+\frac{1}{2}\,x\sin(\log x)+c.$
33. $\frac{1}{2}\,\mathrm{e}^{2x}-\mathrm{e}^x-\frac{1}{2}\,\log\left(\mathrm{e}^{2x}+\mathrm{e}^x+2\right)+\frac{5}{\sqrt{7}}\,\operatorname{arctg}\frac{2\mathrm{e}^x+1}{\sqrt{7}}+c.$
34. $\frac{6}{5}\,(1+x^{1/3})^{5/2}-2(1+x^{1/3})^{3/2}+c.$
35. $\frac{1}{3}\,x^3(x^2+1)^{-3/2}+c.$
36. $2\operatorname{arctg}(x-1)+c.$
37. $\operatorname{tg} x+c.$
38. $\log\left|(x+2)^{1/3}-1\right|-\frac{1}{2}\,\log\left((x+2)^{2/3}+(x+2)^{1/3}+1\right)+\sqrt{3}\,\operatorname{arctg}\frac{2(x+2)^{1/3}+1}{\sqrt{3}}+c.$
39. $\frac{-1}{3}\,(x^2+1)^{-3/2}+c.$
40. $x(1-x^2)^{-1/2}-\arcsin x+c.$
41. $2\sqrt{\mathrm{e}^x-1}-2\operatorname{arctg}\sqrt{\mathrm{e}^x-1}+c.$
42. $5\log|x-1|-3\log|x|+\frac{3}{x}+c.$
43. $\frac{-4}{3}\,(1-\sqrt{x}\,)^{3/2}+2\sqrt{x}+c.$
44. $\csc x-\cot x-\log|\csc x+\cot x|-\log|\sin x|=\csc x-\cot x-\log(1+\cos x)+c.$
45. $\frac{2}{7}\,(x-1)^{7/2}+\frac{4}{5}\,(x-1)^{5/2}+\frac{2}{3}\,(x-1)^{3/2}+c.$
46. $\frac{1}{5}\,\operatorname{tg}^5 x+\frac{2}{3}\,\operatorname{tg}^3 x+\operatorname{tg} x+c.$
47. $\frac{-1}{2(1+x^2)}+\frac{1}{4(1+x^2)^2}+c.$
48. $\frac{1}{4}\,\log|\mathrm{e}^x-2|-\frac{1}{4}\,\log(\mathrm{e}^x+2)+c.$
49. $2\operatorname{arctg}\sqrt{1+x}+c.$
50. $\frac{-3}{2}\,(1-x)^{2/3}+3(1-x)^{1/3}-3\log\left|1+(1-x)^{1/3}\right|+c.$
51. $\frac{1}{5}\,\mathrm{e}^x(\cos 2x+2\sin 2x)+c.$
52. $\frac{1}{3}\,x^3\log x-\frac{1}{9}\,x^3+c.$
53. $\frac{1}{5}\,\cos^5 x-\frac{1}{3}\,\cos^3 x+c.$
54. $\frac{3}{8}\,x+\frac{1}{4}\,\sin 2x+\frac{1}{32}\,\sin 4x+c.$
55. $\frac{1}{3}\,\operatorname{tg}^3 x-\operatorname{tg} x+x+c.$
56. $\frac{1}{2}\,\sec x\operatorname{tg} x+\frac{1}{2}\,\log|\sec x+\operatorname{tg} x|+c.$
57. $\operatorname{tg} x+\sec x+c.$
58. $\frac{-1}{2}\,x\cos(\log x)+\frac{1}{2}\,x\sin(\log x)+c.$
59. $-\frac{\sqrt{1-x^2}}{x}+c.$
60. $\sqrt{1+x^2}+c.$
61. $\operatorname{arctg}\sqrt{\mathrm{e}^{2x}-1}+c.$
62. $\frac{1}{2}\,\mathrm{e}^{2x}-2\mathrm{e}^x+\log\left(\mathrm{e}^{2x}+2\mathrm{e}^x+2\right)+2\operatorname{arctg}(\mathrm{e}^x+1)+c.$
63. $\frac{1}{2}\,x^2+\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+c.$
64. $\frac{-3}{2}\,(1-2x)^{1/3}-3(1-2x)^{1/6}-3\log|(1-2x)^{1/6}-1|+c.$
65. $-\frac{\sqrt{9-x^2}}{9x}+c.$
66. $\frac{-1}{3(x-1)}-\frac{1}{3}\,\log|x-1|+\frac{1}{6}\,\log(x^2+x+1)+\frac{1}{3\sqrt{3}}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+c.$

67. $\frac{1}{m+1}\, x^{m+1} \log x - \frac{1}{(m+1)^2}\, x^{m+1} + c$, if $m \neq -1$; $\frac{1}{2}\,(\log x)^2$ if $m = -1$.
68. $\frac{-1}{3}\, \csc^3 x + \csc x + c$.
69. $\frac{2}{3}\,(-x^{3/2} \cos x^{3/2} + \sin x^{3/2}) + c$.
70. $\frac{1}{2}\, x + \frac{1}{10}\, x \cos(2 \log x) + \frac{1}{5}\, x \sin(2 \log x) + c$.
71. $x(\log x)^3 - 3x(\log x)^2 + 6x \log x - 6x + c$.
72. $\frac{1}{2}\, x^2 (\log x)^2 - \frac{1}{2}\, x^2 \log x + \frac{1}{4}\, x^2 + c$.

Problem 1.2 Find recurrence relations for the following antiderivatives:

$$\begin{array}{llll} i) & H_n = \int \log^n x\, dx, & ii) & I_n = \int \sin^n x\, dx, \\ iii) & J_n = \int x^n \mathrm{e}^{-x}\, dx, & iv) & K_n = \int \frac{dx}{(x^2+1)^n}, \\ v) & L_n = \int \mathrm{tg}^n x\, dx, & vi) & M_n = \int \sec^n x\, dx, \\ vii) & N_n = \int x^n \mathrm{e}^{ax}\, dx, & viii) & P_n = \int x^n \mathrm{e}^{x^2}\, dx. \end{array}$$

Solution: i) $H_n = x \log^n x - n H_{n-1}$; *ii)* $I_n = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$; *iii)* $J_n = -x^n \mathrm{e}^{-x} + n J_{n-1}$; *iv)* $K_n = \frac{1}{2(n-1)} \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)} K_{n-1}$; *v)* $L_n = \frac{1}{n-1}\, \mathrm{tg}^{n-1} x - L_{n-2}$. *vi)* $M_n = \frac{1}{n-1}\, \mathrm{tg}\, x \sec^{n-2} x + \frac{n-2}{n-1} M_{n-2}$; *vii)* $N_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} N_{n-1}$; *viii)* $P_n = \frac{1}{2} x^{n-1} e^{x^2} - \frac{n-1}{2} P_{n-2}$.

Problem 1.3

i) Compute $\displaystyle\int_a^b x\, dx$ using upper and lower sums associated to regular partitions of the interval $[a, b]$.

ii) Do it also for $\displaystyle\int_a^b x^2\, dx$.

Hint: *i)* $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$; *ii)* $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Problem 1.4 Prove and interpret the following identities:

$$i) \qquad \int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x-c)\, dx$$

$$ii) \qquad \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

$$iii) \qquad \int_{-a}^{a} \Big(f(x) - f(-x) \Big)\, dx = 0$$

$$iv) \qquad \left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

$$v) \qquad \int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}.$$

Problem 1.5

i) Prove that, if g is an odd and integrable function on $[-a,a]$, then $\int_{-a}^{a} g = 0$. Apply the result to compute

$$\int_6^{10} \sin[\sin\{(x-8)^3\}]dx\,.$$

ii) Prove that, if h is an even and integrable function on $[-a,a]$, then $\int_{-a}^{a} h = 2\int_0^a h$.

Problem 1.6 Let g be a continuous function and g' continuous on $[a,b]$, such that $g(a)=g(b)$ and $\alpha \le g(x) \le \beta$ for all $x \in [a,b]$. Show that for every integrable f on $[\alpha,\beta]$,

$$\int_a^b f(g(x))g'(x)\,dx = 0.$$

Is it true that $\int_a^b f(g(x))\,dx = 0$?
Hint: Change variables.

Problem 1.7 Let $f,\ g : [-1,1] \to I\!R$ be piecewise continuous functions.

i) Show that

$$\int_{-\pi}^{0} f(\sin x)\ g(\cos x)\,dx = \int_{\pi}^{2\pi} f(\sin x)\ g(\cos x)\,dx.$$

ii) Let f be an odd function. Prove that

$$\int_0^{2\pi} f(\sin x)\,g(\cos x)\,dx = 0.$$

iii) Indicate which of the following integrals are equal to zero:

$$a)\quad \int_0^{2\pi} \mathrm{e}^{|\cos x|}\,\mathrm{arctg}(\sin^5 x)\,dx, \qquad b)\quad \int_0^{2\pi} \mathrm{e}^{\cos x}\mathrm{e}^{-|\sin x|}\,dx,$$

$$c)\quad \int_0^{2\pi} \frac{\sin(\cos^4 x)\sin(\sin x)}{(1+\cos^2 x)(3-\sin^2 x)}\,dx.$$

Problem 1.8 Let f be a periodic function of period T, integrable on $[0,T]$.

i) Prove that for all $a \in I\!R$, we have

$$\int_a^{a+T} f = \int_0^T f.$$

ii) Prove that for all integer n

$$\int_a^b f = \int_{a+nT}^{b+nT} f.$$

Hint: *i*) Separate the integral on $[a, a+T]$ as the sum of integrals on $[a,T]$ and $[T, a+T]$, and use a change of variables in the second integral;
ii) Change variables.

Problem 1.9 Using the two previous problems, prove that if f and g are piecewise continuous functions on $[-1,1]$, and g is a an odd function, we have

$$\int_0^{2\pi} f(\sin x)\, g(\cos x)\, dx = 0.$$

Hint: Apply the change of variables $t = \pi/2 - x$, and use periodicity.

Problem 1.10

i) Prove that if f, g are piecewise continuous functions on $[-1,1]$ and $n > 0$

$$\int_a^{a+2\pi/n} f(\sin nx)\, g(\cos nx)\, dx = 0,$$

being odd at less one of the functions f or g.

ii) What is the value of the same integral on the interval $[a, a+2\pi]$ if $n \in I\!N$?

iii) Indicate which of the following integrals are equal to zero:

$$a) \quad \int_\pi^{2\pi} \cos(\cos 2x)\,\mathrm{arctg}(\sin^5 2x)\, dx, \quad b) \quad \int_{\pi/2}^{5\pi/2} \mathrm{tg}(\cos x)\, dx$$

$$c) \quad \int_{-\pi}^{3\pi} \frac{\sin(\cos^4 x)\sin(\sin x)}{(1+\cos^2 x)(3-\sin^2 x)}\, dx \qquad d) \quad \int_0^{2\pi} \mathrm{tg}\left(\frac{1+\cos x}{2}\right) dx.$$

iv) Prove that if f and g are even functions, then

$$\int_0^{2\pi} f(\sin x)\, g(\cos x)\, dx = 4\int_0^{\pi/2} f(\sin x)\, g(\cos x)\, dx.$$

Problem 1.11 Evaluate the following limits associating them to some definite integral:

$$i) \qquad \lim_{n\to\infty}\left[\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right]$$

$$ii) \qquad \lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right]$$

$$iii) \qquad \lim_{n\to\infty} \frac{\sqrt[n]{\mathrm{e}^2} + \sqrt[n]{\mathrm{e}^4} + \cdots + \sqrt[n]{\mathrm{e}^{2n}}}{n}$$

$$iv) \qquad \lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$v) \qquad \lim_{n\to\infty}\left[\frac{1}{n^2+1} + \frac{1}{n^2+4} + \cdots + \frac{1}{n^2+n^2}\right].$$

Solution: *i)* $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$; *ii)* $\int_0^1 \frac{dx}{1+x} = \log 2$; *iii)* $\int_0^1 \mathrm{e}^{2x} dx = (\mathrm{e}^2 - 1)/2$; *iv)* $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$; *v)* $\int_0^1 \frac{dx}{1+x^2} \cdot \lim_{n\to\infty} \frac{1}{n} = 0$; (also $0 \le \sum_{j=1}^{n} \frac{1}{n^2+j^2} \le \frac{n}{n^2+1} \to 0$).

Problem 1.12 Compute the limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{1/n}.$$

Solution: $\mathrm{e}^{\int_0^1 \log(1+x)dx} = 4/\mathrm{e}$.

Problem 1.13 Evaluate $F(x) = \displaystyle\int_{-1}^{x} f(t)\,dt$ with $x \in [-1,1]$, for the following functions:

i) $f(x) = |x - 1/2|$

ii) $f(x) = |x|\,\mathrm{e}^{-|x|}$

iii) $f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1 \end{cases}$

iv) $f(x) = \begin{cases} x^2 & -1 \le x < 0 \\ x^2 - 1 & 0 \le x \le 1 \end{cases}$

v) $f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ x+1 & 0 < x \le 1 \end{cases}$

vi) $f(x) = \begin{cases} x+2 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \\ -x+2 & 1 \le x \le 2 \end{cases}$

vii) $f(x) = \max\{\sin(\pi x/2), \cos(\pi x/2)\}$.

Solution: i) $F(x) = \begin{cases} -x^2/2 + x/2 + 1 & \text{if } -1 \le x \le 1/2 \\ x^2/2 - x/2 + 5/4 & \text{if } 1/2 \le x \le 1 \end{cases}$;
ii) $F(x) = \begin{cases} (1-x)\mathrm{e}^{x} - 2\mathrm{e}^{-1} & \text{if } x \le 0 \\ (-1-x)\mathrm{e}^{-x} + 2 - 2\mathrm{e}^{-1} & \text{if } x \ge 0 \end{cases}$; *iii)* $F(x) = |x| - 1$;
iv) $F(x) = \begin{cases} (x^3+1)/3 & \text{if } -1 \le x \le 0 \\ -x + (x^3+1)/3 & \text{if } 0 \le x \le 1 \end{cases}$;
v) $F(x) = \begin{cases} 1+x & \text{if } -1 \le x \le 0 \\ 1 + x + x^2/2 & \text{if } 0 \le x \le 1 \end{cases}$; *vi)* $x+1$.

Problem 1.14 Let $F(x) = \displaystyle\int_{a}^{x} f(t)\,dt$ with f integrable.

i) Prove that if $|f| \le M$ on the interval $[\alpha, \beta]$, then $|F(x) - F(y)| \le M|x - y|$ for all $x, y \in [\alpha, \beta]$, implying the (uniformly) continuity of F.

ii) Is F differentiable necessarily? Under what conditions can we say that is differentiable?

Hint: *i)* $F(x) - F(y) = \int_y^x f(t)dt$.

Problem 1.15 Differentiate the following functions:

i) $F(x) = \displaystyle\int_{x^2}^{x^3} \frac{\mathrm{e}^t}{t}\,dt$

ii) $F(x) = \displaystyle\int_{-x^3}^{x^3} \frac{dt}{1+\sin^2 t}$

iii) $F(x) = \displaystyle\int_{3}^{\int_1^x \sin^3 t\,dt} \frac{ds}{1+\sin^6 s + s^2}$

iv) $F(x) = \displaystyle\int_{2}^{\mathrm{e}^{\int_1^{x^2} \mathrm{tg}\sqrt{t}\,dt}} \frac{ds}{\log s}$

v) $F(x) = \displaystyle\int_0^x x^2 f(t)dt,$ with f continuous on $\mathbb{R}$,

vi) $F(x) = \sin\left(\displaystyle\int_0^x \sin\left(\int_0^y \sin^3 t dt\right) dy\right).$

Solution: *i)* $3e^{x^3}/x - 2e^{x^2}/x$; *ii)* $6x^2/(1+\sin^2 x^3)$;
iii)$(\sin x)^3/[1+\sin^6(\int_1^x \sin^3 t dt) + (\int_1^x \sin^3 t dt)^2]$; *iv)* $2x \operatorname{tg}|x| e^{\int_1^{x^2} \operatorname{tg}\sqrt{t}\, dt} / (\int_1^{x^2} \operatorname{tg}\sqrt{t}\, dt)$;
v) $2x \int_0^x f(t)dt + x^2 f(x)$; *vi)* $\cos(\int_0^x \sin(\int_0^y \sin^3 t dt) dy) \sin(\int_0^x \sin^3 t dt)$.

Problem 1.16 Find the points were f attains its absolute maximum and minimum on $[1, \infty)$, where f is:

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t} \right) dt\,,$$

knowing that $\lim_{x\to\infty} f(x) = (\sqrt{\pi} - 1)/2$. Is the absolute minimum of f on $[1, \infty)$ also its absolute minimum on $I\!R$?

Solution: $x = 1$ minimum, $x = 3$ maximum. Yes.

Problem 1.17 Find the tangent line to the graph of $f(x) = \displaystyle\int_{x^2}^{\sqrt{\pi}/2} \operatorname{tg}(t^2)\, dt$ at the point $x = \sqrt[4]{\pi/4}$.

Solution: $y = -\sqrt[4]{4\pi} x + \sqrt{\pi}$.

Problem 1.18 Compute the following limits:

$$i) \quad \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{t^2}\, dt - x}{x^3} \qquad ii) \quad \lim_{x\to 0} \frac{\cos x \displaystyle\int_0^x \sin t^3\, dt}{x^4}.$$

Solution: *i)* $1/3$; *ii)* $1/4$.

Problem 1.19 If the integral $\displaystyle\int_{-1/x}^{x} \frac{dt}{a^2 + t^2}$ does not depend on x, find a without computing the integral.

Solution: $a = \pm 1$.

Problem 1.20 Find the continuous function f verifying $xf(x) = \displaystyle\int_0^x f(t)\, dt$, $f(0) = 1$.

Solution: $f(x) = 1$.

Problem 1.21 If the function g is given by the equation

$$t = \int_0^{|g(t)|^2} \frac{\sin x}{x}\, dx,$$

compute:

i) $g'(t)$ in terms of $g(t)$,

ii) $(g^{-1})'(x)$.

Solution: *i)* $g'(t) = \frac{g(t)}{2\sin(g(t)^2)}$; *ii)* $(g^{-1})'(x) = \frac{2\sin x^2}{x}$.

Problem 1.22 The equation

$$\int_1^{f(x)} e^{-t^2}\, dt - 2x + \log(\cos x) = 0$$

defines a differentiable and one to one function f on the interval $[-1/2, 1/2]$. Find:

i) $f(0),\ f'(0)$ and $(f^{-1})'(1)$,

ii) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{f^{-1}(x+1)}$.

Solution: *i)* $f(0) = 1,\ f'(0) = 2e,\ (f^{-1})'(1) = 1/(2e)$; *ii)* 4e.

Problem 1.23 The equation

$$\int_0^{g(x)} (e^{t^2} + e^{-t^2})\, dt - x^3 - 3 \operatorname{arctg} x = 0$$

defines a differentiable and one to one function g on $\mathbb{R}$. Find:

i) $g(0),\ g'(0)$ and $(g^{-1})'(0)$,

ii) $\displaystyle\lim_{x\to 0} \frac{g^{-1}(x)}{g(x)}$.

Solution: *i)* $g(0) = 0,\ g'(0) = 3/2,\ (g^{-1})'(0) = 2/3$; *ii)* $4/9$.

Problem 1.24

i) Find the explicit formula of a function $f : \mathbb{R} \to \mathbb{R}$ verifying

$$\int_0^x f(t)\, dt = \int_x^1 t^2 f(t)\, dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C.$$

Next, find the value of C.

ii) Do the same problem for the function g:

$$\int_0^x g(t)\, dt = \int_x^1 g(t) \sin t\, dt - \cos x + D.$$

iii) Find $\displaystyle\lim_{x\to 0} \frac{\int_0^{x^2} e^{-t^2} dt}{\int_0^x g(t) dt}$.

Solution: *i)* $f(x) = 2x^{15}$; $C = -1/9$; *ii)* $g(x) = \frac{\sin x}{1+\sin x}$ and $D = \cos 1 + \sec 1 - \operatorname{tg} 1$; *iii)* 2.

Problem 1.25 Let f be a continuous and strictly positive function on the interval $[0, 1]$. Define the following function

$$F(x) = 2 \int_0^x f(t)\, dt - \int_x^1 f(t)\, dt.$$

Prove that F annihilates on the interval $(0, 1)$ and it happens only once .

Hint: Show that F is monotonic and changes sign.

Problem 1.26 Evaluate the following definite integrals, changing the limits of integration when making a change of variables:

$$i)\quad \int_0^{\log 2} \sqrt{e^x - 1}\, dx \qquad ii)\quad \int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx.$$

Solution: *i)* $t = \sqrt{e^x - 1}$; the integral is $2 - \pi/2$; *ii)* $t = \sqrt{x^2-1}$ (or $x = \sec t$); the integral is $\sqrt{3} - \pi/3$.

Problem 1.27 Consider the functions

$$f(x) = \int_0^x e^{t^2 - x^2}\, dt, \qquad g(x) = \int_0^x e^{t^2}\, dt, \qquad h(x) = \frac{e^{x^2}}{2x},$$

i) Evaluate $\lim_{x\to\infty} \frac{g(x)}{h(x)}$;

ii) Express f in terms of g and h, and express f' in terms of f;

iii) Prove, using the previous parts, that $\lim_{x\to\infty} xf(x) = 1/2$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$.

Solution: *i)* 1; *ii)* $f(x) = g(x)/(2xh(x)),\ f'(x) = -2xf(x) + 1$.

Problem 1.28 Evaluate the following limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n^2-1}{(x^2+1)(1+n^2)} e^{-x^4/n}\, dx.$$

Solution: $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$.

Problem 1.29 Analyze the convergence of the following improper integrals:

$$i)\ \int_{-\infty}^{\infty} e^{-x}\, dx \qquad ii)\ \int_{-\infty}^{\infty} e^{-x^2}\, dx \qquad iii)\ \int_1^{\infty} e^{-x} x^p\, dx$$

$$iv)\ \int_0^a e^{1/x} x^p\, dx \qquad v)\ \int_0^1 \log x\, dx \qquad vi)\ \int_1^{\infty} \frac{dx}{x^\alpha \sqrt{1+x^2}}$$

$$vii)\ \int_0^1 x^p (1-x)^q\, dx \qquad viii)\ \int_1^{\infty} \left(\frac{1}{\sqrt{x}} - \operatorname{arctg} \frac{1}{\sqrt{x}} \right) dx \qquad ix)\ \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+4}\, dx$$

$$x)\ \int_1^2 \frac{\log t + t - 1}{(t-1)^{3/2}}\, dt \qquad xi)\ \int_0^{1/2} \frac{e^{\arcsin x}(5 - \sin x)}{x(\log x)^2 (5 + \sin x)}\, dx \qquad xii)\ \int_0^{\infty} \frac{dx}{x^2+a^2}.$$

Solution: *i)* D; *ii)* C; *iii)* C $\forall p$; *iv)* D $\forall p$, if $a > 0$; C $\forall p$, if $a \le 0$; *v)* C; *vi)* C $\forall \alpha > 0$; *vii)* C $\forall p, q > -1$; *viii)* C; *ix)* C $\forall a \in \mathbb{R}$; *x)* C; *xi)* C; *xii)* C $\forall a \neq 0$.

Problem 1.30

i) Analyze the convergence of the integral

$$\int_0^1 \log x \, \log(x+1)\, dx.$$

ii) Compute its value knowing that $\displaystyle\int_0^1 \frac{\log(x+1)}{x}\, dx = \pi^2/12.$

iii) Prove (without calculator!) that $2\log 2 + \pi^2/12 > 2$.

Solution: i) The integral converges; *ii*) and is equal to $2 - 2\log 2 - \pi^2/12$.

Problem 1.31 Using the value of the integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$, prove that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \alpha t}{t}\, dt = \begin{cases} 1 & \text{if} \quad \alpha > 0 \\ 0 & \text{if} \quad \alpha = 0 \\ -1 & \text{if} \quad \alpha < 0 \end{cases}.$$

With this result and the formula

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

prove also that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin t\, \cos xt}{t}\, dt = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 1/2 & \text{if} \quad |x| = 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}.$$

Hint: It is enough to prove it for $x \geq 0$, because the integrand and the solution are even functions on x.

Problem 1.32 Let the integral $I_k = \displaystyle\int_{-\infty}^{\infty} x^k \mathrm{e}^{-x^2}\, dx.$

i) Prove that I_k is convergent for all $k \in \mathbb{N}$.

ii) Evaluate the integral if k is odd.

iii) Find a recurrence relation for I_{2n}.

iv) Use the previous formula to show, knowing that $I_0 = \sqrt{\pi}$, that

$$I_{2n} = \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\cdots \frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}.$$

Solution: ii) $I_{2n+1} = 0$; *iii*) $I_{2n} = (n - 1/2) I_{2n-2}$.

Problem 1.33 Given the integral

$$I_p = \int_c^1 \frac{t^p}{\sqrt[4]{t^3(1-t)}}\,dt, \qquad 0 \le c < 1.$$

i) Analyze its convergence.

ii) Deduce a relation between I_p and I_{p-1}.

Hint: *ii*) Do integration by parts, differentiating $t^{p-3/4}$, to obtain

$$I_p = \frac{4}{3}c^{p-3/4}(1-c)^{3/4} + \Big(\frac{4p}{3} - 1\Big)(I_{p-1} - I_p).$$

Solution: i) If $c = 0$, I_p converges for $p > -1/4$; if $c > 0$ converges always; *ii*) the relation is

$$I_p = \frac{1}{p}c^{p-3/4}(1-c)^{3/4} + \Big(1 - \frac{3}{4p}\Big)I_{p-1}, \qquad p > 3/4.$$

Problem 1.34 Prove that the following improper integrals converge only for the given values of the parameter:

i) $\displaystyle\int_0^{\pi/2} \log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{dx}{\cos x}$, with $|s| \le 1$.

ii) $\displaystyle\int_0^{\infty} \log\left(1+\frac{a^2}{x^2}\right)dx$, with $a \in I\!\!R$.

iii) $\displaystyle\int_0^1 \frac{x^p-1}{\log x}dx$, with $p > -1$.

Problem 1.35 Consider the improper integral

$$\int_0^{\infty}\Big(\frac{1}{\sqrt{1+x^2}} - \frac{\alpha}{x+1}\Big)dx, \qquad \alpha > 0.$$

i) Prove that this integral only converges for a value of the parameter α and find that value.

ii) Find an antiderivative of the function $(1+x^2)^{-1/2}$.

iii) Evaluate the improper integral for the computed value of α.

Solution: i) $\alpha = 1$; *ii*) $\log(x+\sqrt{1+x^2})$; *iii*) $\log 2$.

Problem 1.36 If we define the following function for $x > 0$

$$f(x) = \int_{1/x}^{x^2} t^{\alpha}\mathrm{e}^{-t^2}dt.$$

i) Find the tangent line to the graph of f at $x = 1$.

ii) Find the horizontal and vertical asymptotes of f for the different values of α.

Solution: i) $y = 3(x-1)/\mathrm{e}$; *ii*) if $\alpha > -1$, there is HA, $y = \int_0^{\infty} t^{\alpha}\mathrm{e}^{-t^2}dt$, there is not VA; if $\alpha \le -1$, there is not HA, there is VA, $x = 0$.

Problem 1.37 Let f be continuous on $[0,1]$, compute the limit

$$\lim_{x\to 0} x\int_{x^2}^{1}\frac{f(t)}{t}\,dt.$$

Hint: If the integral converges, the limit is zero; if it diverges, apply L'Hôpital. Alternative form: if $f=1$, the integral is explicit; if f is continuous on $[0,1]$, f is bounded.

Solution: 0.

Problem 1.38 Let f be a continuous function such that $\lim\limits_{x\to\infty} f(x)=l$,

i) Prove that for all $r>0$ fixed, we have $\lim\limits_{x\to\infty}\dfrac{1}{2r}\int_{x-r}^{x+r} f(t)\,dt = l$.

ii) If we also have $l>0$ and $\sum\limits_{k=0}^{n} f(k)\neq 0$ for all $n\in I\!N$, compute $\lim\limits_{n\to\infty}\dfrac{\int_0^n f(t)\,dt}{\sum\limits_{k=0}^{n} f(k)}$.

Hint: *ii)* Use Stolz test and part *i)*.

Solution: ii) 1.

Problem 1.39 Obtain the power series of $f(x)=\operatorname{arctg} x$, integrating term by term the power series of $f'(x)$.

Solution: $\operatorname{arctg} x = \sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1\le x\le 1$.

Problem 1.40 Prove the following identities:

i) $\lim\limits_{b\to+\infty}\displaystyle\int_0^{\infty} \mathrm{e}^{-ax}\sin bx\,dx = 0 \qquad (a>0),$

ii) $\displaystyle\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi,$

iii) $\displaystyle\int_0^{\infty} t^n \mathrm{e}^{-t}dt = n! \qquad (n\in I\!N).$

Hint: *i)* Compute the integral using parts twice; *ii)* use a change of variables transforming the interval $[a,b]$ onto $[-1,1]$; *iii)* use a recurrence relation already proven in a previous problem.

Problem 1.41 Consider isosceles triangles with base the line segments $(n, n+1/2^n)$ and height 1, where $n\in I\!N$ $(n\ge 0)$. Let f be the function whose graph is the polygonal made by the x axis and the triangles. Show that $\displaystyle\int_0^{\infty} f = 1$.

Problem 1.42 Given the function $f(x)=\begin{cases}(x-1)^2 & 0\le x\le 1\\ 0 & x\ge 1\end{cases}$, define the sequence given by $f_n(x)=nf(nx)$. Evaluate

$$i)\quad \lim_{n\to\infty}\int_0^1 f_n(x)\,dx, \qquad ii)\quad \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

Explain what is happening.

Solution: i) 1/3; *ii)* 0.

Problem 1.43

i) Analyze the convergence of the improper integral $\displaystyle\int_1^\infty \frac{e^x}{x^x}\,dx$ by comparing with the corresponding series.

ii) Analyze the convergence of the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\log n)^{\log n}}$ by comparing with the corresponding integral. Apply a change of variables to this integral to transform it into the previous one.

iii) Analyze the convergence of the series

$$a) \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} \quad \alpha > 0, \qquad b) \quad \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log(\log n)}}.$$

Hint: *iii.b*) Make a change of variables and use the identity $t^\alpha = e^{\alpha \log t}$, for $\alpha, t > 0$.

Solution: i) C, use root test ; *iii.a*) C, $\forall\, \alpha > 1$; *iii.b*) D, because $\lim\limits_{t\to\infty} e^{t-\log^2 t} = \infty$.

Problem 1.44 To study the convergence of the integral $\displaystyle\int_0^\infty (-1)^{[x^2]}\,dx$, consider the limit

$$\lim_{R\to\infty} \int_0^R (-1)^{[x^2]}\,dx = \lim_{R\to\infty} \left(\int_0^{\sqrt{[R^2]}} (-1)^{[x^2]}\,dx + \int_{\sqrt{[R^2]}}^R (-1)^{[x^2]}\,dx \right),$$

and analyze each limit separately.

i) Show that $\lim\limits_{R\to\infty} (R - \sqrt{[R^2]}) = 0$.

ii) Let $M \in \mathbb{N}$, use the change of variables $x^2 = t$ to prove the formula

$$\int_0^{\sqrt{M}} (-1)^{[x^2]}\,dx = \sum_{n=1}^{M} (-1)^{n-1} \left(\sqrt{n} - \sqrt{n-1} \right).$$

iii) Conclude the convergence of the integral.

iv) Is it absolutely convergent?

Problem 1.45

i) Evaluate $\displaystyle\int_1^n \log x\,dx$. Compare the previous integral with its upper and lower sums for the partition $P = \{1, 2, \ldots, n\}$. Deduce from the previous comparison the following inequality:

$$(n-1)! \le n^n e^{-n+1} \le n!\,.$$

ii) Evaluate $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n}$.

Solution: ii) $1/e$.

Problem 1.46 Find the area enclosed by the following curves:

i) $y = x^2,\ y = (x-2)^2,\ y = (2-x)/6,$

ii) $x^2 + y^2 = 1,\ x^2 + y^2 = 2x,$

iii) $y = (1-x)/(1+x),\ y = (2-x)/(1+x),\ y = 0,\ y = 1,$

iv) loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Solution: *i)* 25/48; *ii)* $2\pi/3 - \sqrt{3}/2$; *iii)* $\log 2$; *iv)* $8(b-a)^{5/2}/15$.

Problem 1.47 Find the area enclosed by the following curves given in parametric and polar coordinates:

i) loop: $x = t^2 + 1,\ y = t(t^2 - 4), \quad -2 \le t \le 2,$

ii) cycloid: $x = a(t - \sin t),\ y = a(1 - \cos t), \quad 0 \le t \le 2\pi$, and axis x,

iii) spiral of Archimedes: $r = a\theta, \quad 0 \le \theta \le 2\pi$, and axis x,

iv) three-leaved rose: $r = a\cos 3\theta$,

v) lemniscate: $r = a\sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4$.

Solution: *i)* 256/15; *ii)* $3\pi a^2$; *iii)* $4a^2\pi^3/3$; *iv)* $\pi a^2/4$; *v)* $a^2/4$.

Problem 1.48 Find the area between the graph of the function $f(x) = \dfrac{x^2-4}{x^2+4}$ and its asymptote.

Solution: The asymptote is $y = 1$ and the area 4π.

Problem 1.49 Let A be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Compute the area of A and the revolution volume obtained by rotating A about the horizontal axis.

Solution: The area is 1/3 and the volume $3\pi/10$.

Problem 1.50 Evaluate the volumes formed by revolving the following regions about the x axis:

i) $0 \le y \le 1 + \sin x, \quad 0 \le x \le 2\pi,$

ii) $x^2 + (y-2a)^2 \le a^2$, the graph is the torus,

iii) $R^2 \le x^2 + y^2 \le 4R^2$, the graph is an spherical ring,

iv) the surface bounded by the curves $y = \sin x$ and $y = x$, with $x \in [0, \pi]$,

v) $x = t - \sin t,\ 0 \le y \le 1 - \cos t, \quad 0 \le t \le 2\pi$.

Solution: *i)* $3\pi^2$; *ii)* $4\pi^2 a^3$; *iii)* $28\pi R^3/3$; *iv)* $\pi^4/3 - \pi^2/2$; *v)* $5\pi^2$.

Problem 1.51

i) Compute the volumes formed by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ about the x and y axes.

ii) Compute the volume of the solid with base the previous ellipse and whose perpendicular sections to the x axis are isosceles triangles of height 2.

Solution: *i*) $4\pi ab^2/3$ and $4\pi a^2b/3$ respectively; *ii*) πab.

Problem 1.52 Find the length of the following graphs:

i) catenary: $y = \mathrm{e}^{x/2} + \mathrm{e}^{-x/2}, \quad 0 \le x \le 2,$

ii) cycloid: $x(t) = a(t - \sin t),\ y(t) = a(1 - \cos t), \quad 0 \le t \le 2\pi,$

iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4,$

iv) tractrix: $y = a\log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}, \quad a/2 \le x \le a,$

v) cardioid: $r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi,$

vi) circular helix : $x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) = bt, \quad 0 \le t \le 2\pi.$

Solution: *i*) $\mathrm{e} - 1/\mathrm{e}$; *ii*) $8a$; *iii*) 48; *iv*) $a\log 2$; *v*) 8; *vi*) $2\pi\sqrt{a^2 + b^2}$.

2 INTEGRATION IN $I\!R^n$

Problem 2.1 Let $f : Q = [0,1] \times [0,1] \longrightarrow I\!R$ be given by $f(x,y) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$. Prove that f is integrable and $\displaystyle\int_Q f = \frac{1}{2}$.

Problem 2.2 Let $f : A \subset I\!R^2 \longrightarrow I\!R$ be integrable on A and let $g : A \longrightarrow I\!R$ with $g = f$ except for a finite number of points. Prove that g is integrable on A and $\displaystyle\int_A g = \int_A f$.

Problem 2.3

i) Prove, without computing the integral, that

$$4\pi \le \int_D (x^2 + y^2 + 1)\,dx\,dy \le 20\pi\,,$$

where D is the disk of radius 2 centered at the origin.

ii) Let A be the square $[0,2] \times [1,3]$ and let $f(x,y) = x^2y$. Prove, without computing the integral, that

$$0 \le \int_A f(x,y)\,dx\,dy \le 48\,.$$

iii) Improve this last estimation and prove that

$$3 \le \int_A f(x,y)\,dx\,dy \le 25\,.$$

Hint: Use a partition of A consisting of four equal squares.

Problem 2.4 Approximate in the following cases, by means of upper and lower sums, the integral $\int_R f(x,y)\, dA$, where $R = [0,4] \times [0,2]$. Use a partition consisting of eight equal squares. Compute also the integral exactly and compare the results.

$$\begin{array}{ll} i) \;\; f(x,y) = x + y & b) \;\; f(x,y) = xy \\ ii) \;\; f(x,y) = x^2 + y^2 & d) \;\; f(x,y) = 1/[(x+1)(y+1)]. \end{array}$$

Solution: *i)* $16 < I < 32$, $(I = 24)$; *ii)* $6 < I < 30$, $(I = 16)$; *iii)* $32 < I < 80$, $(I = 160/3)$; *iv)* $77/72 < I < 25/8$, $(I = \log 3 \log 5)$.

Problem 2.5 Let f be the function defined on the square $Q = [0,1] \times [0,1]$:

$$f(x,y) = \begin{cases} 1 - x - y & \text{if } x + y \leq 1 \\ 0 & \text{if } x + y \geq 1 \end{cases}$$

Sketch the graph of f over Q and evaluate $\int_Q f$.

Solution: 1/6.

Problem 2.6 *(Cavalieri's Principle)* Let $A, B \subset I\!R^3$ be two regions. If we define the sections $A_c = \{(x,y) \in I\!R^2 \,/\, (x,y,c) \in A\}$ and $B_c = \{(x,y) \in I\!R^2 \,/\, (x,y,c) \in B\}$. Suppose that A_c and B_c have the same area for each value of c. Prove that A and B have the same volume.

Problem 2.7 From the previous problem it follows that two pyramids with the same base and height have the same volume. Find that volume by integrating.

Solution: $V = Ah/3$, where A = area of the base.

Problem 2.8 We call *cone* the three–dimensional picture obtained by joining all the points of a planar region S to a point located out of the S plane. Let A be the area of S and h the height of the cone, show that:

i) The area of the section of a parallel plane to the base at a distance t from the vertex is $(t/h)^2 A$, for $0 \leq t \leq h$.

ii) The volume of the cone is $Ah/3$.

Problem 2.9 Prove that the following regions of $I\!R^2$ have null measure (area, in this case):

i) $S = \{(x,y) \in I\!R^2 \,/\, |x| + |y| = 1\}$,

ii) $U = \{(x,y) \in I\!R^2 \,/\, x^2 + y^2 = 1\}$,

iii) the graph of $f : [a,b] \longrightarrow I\!R$, continuous, $G = \{(x, f(x)) \,/\, x \in [a,b]\}$.

Problem 2.10 Let $f : A \subset I\!R^2 \longrightarrow I\!R$ be a function greater than or equal to 0 and integrable on A, with $\int_A f = 0$.

i) Let $A_m = \{\mathbf{x} \in A \,/\, f(\mathbf{x}) > 1/m\}$; prove that A_m has null measure (area).

ii) Deduce that the region where $f(\mathbf{x}) \neq 0$ has null measure.

Are these results true if $A \subset I\!R^n$?

Problem 2.11 Identify the type of region and change the order of integration for the following integrals:

$$i) \quad \int_0^3 \int_{4x/3}^{\sqrt{25-x^2}} f(x,y)dy\,dx \qquad\qquad ii) \quad \int_0^1 \int_0^y f(x,y)dx\,dy$$

$$iii) \quad \int_0^{\pi/2} \int_{-\sin(x/2)}^{\sin(x/2)} f(x,y)dy\,dx \qquad iv) \quad \int_1^{\mathrm{e}} \int_0^{\log x} f(x,y)dy\,dx.$$

Solution: i) $\{0 \le y \le 4,\ 0 \le x \le 3y/4\} \cup \{4 \le y \le 5,\ 0 \le x \le \sqrt{25-y^2}\}$; *ii)* $\{0 \le x \le 1,\ x \le y \le 1\}$; *iii)* $\{-\sqrt{2}/2 \le y \le 0,\ -2\arcsin y \le x \le \pi/2\} \cup \{0 \le y \le \sqrt{2}/2,\ 2\arcsin y \le x \le \pi/2\}$; *iv)* $\{0 \le y \le 1,\ \mathrm{e}^y \le x \le \mathrm{e}\}$.

Problem 2.12

i) Over the region $R = \{(x,y) \in I\!R^2 \,/\, x^2 + (y-1)^2 \le 1,\ x \ge 0\}$, consider the functions $f(x,y) = \dfrac{1}{\sqrt{1-x^2}}$ and $g(x,y) = \sin(y-1)$. Apply Fubini's Theorem to $\int_R f$ and $\int_R g$ in the two possible ways. Evaluate the integrals for the more convenient order.

ii) Find the integral over the same region of the function $h(x,y) = \dfrac{\sqrt{2y^2+x^2}}{y}$.

Solution: i) $\int_R f = 2,\ \int_R g = 0$; *ii)* $\int_R h = 1 + \pi/2$.

Problem 2.13 Find the value of the integral $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx\,.$

Solution: 2.

Problem 2.14 Prove the identities

$$i) \quad \int_0^x \int_0^t F(u)\,du dt = \int_0^x (x-u)F(u)\,du$$

$$ii) \quad \int_0^x \int_0^v \int_0^u f(t)\,dt du dv = \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt.$$

Problem 2.15 Evaluate $\displaystyle\int_D (x^2+y)\,dx\,dy$, where $D = \{\,(x,y) \in I\!R^2\ :\ |x|+|y| \le 1\,\}$.
Hint: Transform the integral into an integral on the region of D in the first quadrant.

Solution: 1/3.

Problem 2.16 Evaluate $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dx\,dy$, where $f(x,y) = \max(|x|,|y|)$.

Solution: 2/3.

Problem 2.17 Compute the following volumes:

i) volume of intersection of the cylinder $x^2 + y^2 \le 4$ and the ball $x^2 + y^2 + z^2 \le 16$;

ii) volume of intersection of the cylinders $x^2 + y^2 \le 4$ and $x^2 + z^2 \le 4$;

iii) volume of the solid bounded by the six cylinders $z^2 = y$, $z^2 = 2y$, $x^2 = z$, $x^2 = 2z$, $y^2 = x$ and $y^2 = 2x$;

iv) volume of the solid bounded by the cones $z = 1 - \sqrt{x^2 + y^2}$ and $z = -1 + \sqrt{x^2 + y^2}$;

v) volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$ in $z \ge 0$;

vi) volume of the region bounded by $x^2 + y^2 + z^2 \le 2$, $x^2 + y^2 \le z$ and $z \le 6/5$;

vii) volume of the region bounded by the surfaces $z = x^2 + y^2$, $z = 2(x^2 + y^2)$, $y = x$ and $y^2 = x$.

Solution: *i)* $32\pi(8 - 3\sqrt{3})/3$; *ii)* $128/3$; *iii)* $1/7$; *iv)* $2\pi/3$; *v)* 8π; *vi)* $493\pi/750$; *vii)* $3/35$.

Problem 2.18 Let the following mapping be defined by $\begin{cases} x = u + v \\ y = v - u^2 \end{cases}$. Evaluate:

i) the Jacobian $J(u, v)$;

ii) the image S on the xy plane of the triangle T on the uv plane of vertices (0,0), (2,0) and (0,2);

iii) the area of S;

iv) the integral $\displaystyle\int_S (x - y + 1)^{-2} dxdy$.

Solution: *i)* $1 + 2u$; *iii)* $14/3$; *iv)* $2 + (\pi - 6\,\mathrm{arctg}(5/\sqrt{3}))\sqrt{3}/9$.

Problem 2.19 Use a linear mapping to compute $\displaystyle\int_S (x - y)^2 \sin^2(x + y)\, dxdy$, where S is the parallelogram of vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution: $\pi^4/3$.

Problem 2.20 Evaluate $\displaystyle\int_D (y - x)\, dxdy$, where D is the region of the plane bounded by $y = x + 1$, $y = x - 3$, $y = (7 - x)/3$ and $y = 5 - x/3$.

Solution: -8.

Problem 2.21 Find the following areas:

i) area of the region $A = \{(x, y) \in \mathbb{R}^2 \,:\, x, y > 0,\ a^2 y \le x^3 \le b^2 y,\ p^2 x \le y^3 \le q^2 x,\ \}$, where $0 < a < b$ and $0 < p < q$.

ii) area bounded by the curves $xy = 4$, $xy = 8$, $xy^3 = 5$ and $xy^3 = 15$.

Solution: *i)* $(b - a)(q - p)/2$; *ii)* $2 \log 3$.

Problem 2.22 Find the integral of the function

$$f(x,y) = \frac{y^4}{b^4\Big(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\Big)\Big(1+\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\Big)} + xy^2$$

over the region $D = \Big\{ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1 \Big\}$, where a and b are positive constants.

Solution: $3\pi ab(1-\log 2)/8$.

Problem 2.23 Find the integral of the function

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}\, \mathrm{e}^{\sqrt{x^2+y^2}}$$

over the regions $E = \{\, x^2 + (y-1)^2 \le 1 \,\}$ and $H = \{\, x^2 + (y-1)^2 \le 1,\ x \ge 0 \,\}$.

Solution: 0, 2.

Problem 2.24 Evaluate $\displaystyle\int_D \frac{dx\,dy}{xy}$, where D is the plain region bounded by the curves

$$x^2+y^2 = ax\,,\ x^2+y^2 = a'x\,,\ x^2+y^2 = by\,,\ x^2+y^2 = b'y\,,$$

where $0 < a < a'\,,\ 0 < b < b'$.
Hint: Change variables appropriately, so the new region is the rectangle $[a,a'] \times [b,b']$.

Solution: $\log(a'/a)\log(b'/b)$.

Problem 2.25 Evaluate the integral $\displaystyle\int_S \frac{x\,dx\,dy}{4x^2+y^2}$, where S is the region on the first quadrant bounded by the coordinate axes and the ellipses $4x^2+y^2 = 16,\ 4x^2+y^2 = 1$.

Solution: 3/4.

Problem 2.26 Let $f(x,y)$ be an odd function on the x variable, that is, $f(-x,y) = -f(x,y)$, and integrable on the region $D \subseteq \mathbb{R}^2$ that is symmetric with respect to the x variable (that is, $(x,y) \in D$ if and only if $(-x,y) \in D$). Prove that if f is integrable on D, then $\int_D f = 0$.

Problem 2.27 Evaluate the volume of the solid bounded by the surfaces $y = z^2,\ 2y = z^2,$ $z = x^2,\ 2z = x^2,\ x = y^2,\ 2x = y^2$.
Hint: Make a change of variables so the new region of integration is the cube $[1,2]^3$. Find the Jacobian of the inverse change.

Solution: 1/7.

Problem 2.28 Let R be the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$, evaluate

$$i)\ \int_R \sqrt{x^2+y^2+z^2}\, dxdydz\,, \quad ii)\ \int_R \sqrt{9-x^2-y^2}\, dxdydz\,.$$

Solution: *i)* $27\pi(2\sqrt{2}-1)/2$; *ii)* $54\pi - 81\pi^2/8$.

Problem 2.29 Evaluate $\int_W f(x,y,z)\,dxdydz$, in the following cases:

i) $f(x,y,z) = \mathrm{e}^{-(x^2+y^2+z^2)^{3/2}}$, and W is the region under the sphere $x^2+y^2+z^2=9$ and over the cone $z=\sqrt{x^2+y^2}$.

ii) $f(x,y,z) = z\,\mathrm{e}^{x^2+y^2+z^2}$, and $W = \{\,x^2+y^2 \le z^2,\ 0 \le z \le 1\,\}$.

iii) $f(x,y,z) = \sqrt{1-x^2-y^2} + \dfrac{xyz^3}{1+z^2}$, and $W = \{\,x^2+y^2+z^2 \le 1,\ x^2+y^2 \le a^2\,\}$, if $0 < a < 1$.

Solution: *i*) $\pi(2-\sqrt{2})(1-\mathrm{e}^{-27})/3$; *ii*) $\pi(\mathrm{e}-1)^2/4$; *iii*) $\pi a^2(2-a^2)$.

Problem 2.30 Evaluate the volume of the solid bounded by the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. Analyze the particular instance $a=b=c=r$.

Solution: *i*) $4\pi abc/3$.

Problem 2.31

i) Evaluate the area of the region $D = \{\,x = r\cos^3 t,\ y = r\sin^3 t,\ 0 \le r \le 1, 0 \le t \le \pi/2\} = \{\,x^{2/3}+y^{2/3} \le 1,\ x,\,y \ge 0\}$.

ii) Find the center of mass of D if its mass density is 1.

Solution: *i*) $3\pi/32$; *ii*) $x_{CM} = y_{CM} = 256/(315\pi)$.

Problem 2.32 The first octant of the ball $x^2+y^2+z^2 \le c^2$ is sliced with the plane $\dfrac{x}{a}+\dfrac{y}{b}=1$, $0 < a, b \le c$. Find the mass of each of the resulting solids knowing that the density is $\rho(x,y,z) = z$.

Solution: One of them is $ab(a^2+b^2-6c^2)/24 + \pi c^2/12$.

Problem 2.33 Find the mass of the sheet corresponding to the portion of the first quadrant of the circle $x^2+y^2 \le 4$, if the density at (x,y) is proportional to the distance between the point and the center of the circle.

Solution: $4\pi\alpha/3$, where α is the proportionality constant.

Problem 2.34 The temperature at points in the cube $[-1,1]^3$ is proportional to the square of its distance from the origin.

i) What is the average temperature?

ii) At which points of the cube is the temperature equal to the average temperature?

Solution: *i*) α, where α is the proportionality constant; *ii*) on the unit sphere.

Problem 2.35 Find the center of mass of the hemispherical region of radius R if the density at each point is the square of the distance of the point to the center.

Solution: $(0,0,5R/12)$.

Problem 2.36 An ice cream cone is made by a cone of angle α and an ice cream hemisphere of radius R. The cornet and the ice cream have constant densities ρ_c and ρ_h respectively. Determine the value of ρ_c/ρ_h such that the center of mass of the ice cream is located on the plane that separates the ice cream and the cornet.

Solution: $3\,\mathrm{tg}^2\,\alpha$.

Problem 2.37 Evaluate

$$i)\ \int_0^1\int_0^1\cdots\int_0^1 (x_1^2+x_2^2+\cdots+x_n^2)\,dx_1dx_2\ldots dx_n,$$

$$ii)\ \int_0^1\int_0^1\cdots\int_0^1 (x_1+x_2+\cdots+x_n)^2\,dx_1dx_2\ldots dx_n.$$

Solution: *i)* $n/3$; *ii)* $(3n^2+n)/12$.

Problem 2.38 Let the function

$$I(p,r)=\int_R \frac{dxdy}{(1+x^2+y^2)^p}\,,$$

where R is the disk of radius r centered at the origin. Find the values of p for which $I(p,r)$ has finite limit when $r\to\infty$.

Solution: $\pi p^{2-2p}/(p-1)$ if $p>1$.

Problem 2.39

i) Evaluate the integral $\int_{D_R} \mathrm{e}^{-(x^2+y^2)}dxdy$, where D_R is the disk of radius R centered at the origin.

ii) Let $Q_{a,b}$ be the rectangle $[-a,a]\times[-b,b]$, prove the estimation

$$\int_{D_{r_1}} \mathrm{e}^{-(x^2+y^2)}dxdy \le \int_{Q_{a,b}} \mathrm{e}^{-(x^2+y^2)}dxdy \le \int_{D_{r_2}} \mathrm{e}^{-(x^2+y^2)}dxdy,$$

for certain r_1 and r_2.

iii) Taking the limit $a,\,b\to\infty$, prove the formula $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}dx=\sqrt{\pi}$.

Solution: *i)* $\pi(1-\mathrm{e}^{-R^2})$; *ii)* $r_1=\min\{a,\,b\},\ r_2=\sqrt{a^2+b^2}\,$.

Problem 2.40 Let f be a continuous function, find $F'(t)$ in the cases

$$i)\ F(t)=\int_0^t\int_0^t\int_0^t f(xyz)\,dxdydz.$$

$$ii)\ F(t)=\iiint_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)\,dxdydz,$$

Hint: *ii)* Use spherical coordinates.

Solution: *i)* $3\int_0^t\int_0^t f(txy)dxdy$; *ii)* $4\pi t^2 f(t^2)$.

3 INTEGRALS DEPENDING ON A PARAMETER

Problem 3.1 Let $f : [a,b] \times [c,d] \longrightarrow I\!R$ be a continuous function with $\dfrac{\partial f}{\partial y}$ continuous. Define $F : [a,b] \times [c,d] \longrightarrow I\!R$ as

$$F(x,y) = \int_a^x f(t,y)\, dt\,.$$

i) Evaluate $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$.

ii) Find the derivative of $G(x) = F(g(x), x)$, being g differentiable. Which are the domain and the range of g?

Solution: ii) $G'(x) = f(g(x), x)g'(x) + \int_a^{g(x)} \frac{\partial f}{\partial y}(t,x)dt.$

Problem 3.2 Find the derivatives of the following functions

$$i) \quad F(y) = \int_{y^2}^{y} \sin(x^2+y^2)dx, \quad ii) \quad G(y) = \int_y^{y^2} \mathrm{e}^{-x^2 y}dx, \quad iii) \quad H(y) = \int_0^\infty \mathrm{e}^{-x^2}\cos(yx)dx.$$

Solution: i) $\sin(2y^2) - 2y\sin(y^4+y^2) + 2y\int_{y^2}^{y}\cos(x^2+y^2)dx$; *ii)* $2y\mathrm{e}^{-y^5} - \mathrm{e}^{-y^3} - \int_y^{y^2} x^2\mathrm{e}^{-x^2y}dx$; *iii)* $-\int_0^\infty x\mathrm{e}^{-x^2}\sin(xy)dx.$

Problem 3.3 Let the function

$$F(x) = \int_0^1 \frac{(\log(1-xt))^2}{t}\, dt.$$

i) Find the values of x such that $F(x)$ is defined (that is, the integral converges).

ii) Evaluate $F'(x)$ and the resulting integral.

iii) Analyze the increasing and decreasing intervals of F.

Solution: i) $(-\infty, 1]$; *ii)* $F'(x) = (\log(1-x))^2/x$; *iii)* F decreases on $(-\infty, 0)$ and increases on $(0,1)$.

Problem 3.4 Let $F, G : I\!R^+ \longrightarrow I\!R$ be defined as

$$F(x) = \left(\int_0^x \mathrm{e}^{-t^2}dt\right)^2 \qquad \text{and} \qquad G(x) = \int_0^1 \frac{\mathrm{e}^{-x^2(1+t^2)}}{1+t^2}dt\,.$$

Prove that:

i) $F'(x) + G'(x) = 0$, for all $x \in I\!R$.

ii) $F(x) + G(x) = \pi/4$, for all $x \in I\!R$.

iii) Deduce that $\displaystyle\int_0^\infty \mathrm{e}^{-x^2}dx = \sqrt{\pi}/2$.

Problem 3.5 Evaluate $F(s) = \int_0^\infty \mathrm{e}^{-x} \sin(sx)\, dx$, and, from it, compute

$$G(s) = \int_0^\infty x\mathrm{e}^{-x} \cos(sx)\, dx.$$

Solution: $F(s) = \frac{s}{1+s^2}$, $G(s) = \frac{1-s^2}{(1+s^2)^2}$.

Problem 3.6 Let $F(\alpha) = \displaystyle\int_0^\infty \frac{\mathrm{e}^{-\alpha x} - \mathrm{e}^{-x}}{x}\, dx.$

i) Analyze the convergence of the integral.

ii) Evaluate $F'(\alpha)$ explicitly and, from it, compute $F(\alpha)$.

iii) From the iterated derivatives of $F^{k)}(\alpha)$, compute

$$\int_0^\infty x^n \mathrm{e}^{-x} dx.$$

Solution: i) It converges for $\alpha > 0$; *ii)* $F'(\alpha) = -1/\alpha$; $F(\alpha) = -\log\alpha$; *iii)* $n!$.

Problem 3.7 Prove that if h is a differentiable function on $[0, \infty)$, such that $\lim_{x\to\infty} h(x) = 0$ and if it is possible to differentiate under the integral sign, then, for all a, $b > 0$,

$$\int_0^\infty \frac{h(ax) - h(bx)}{x}\, dx = h(0)\log(b/a).$$

Problem 3.8 Obtain explicitly the following functions by differentiating and then computing the integral with respect to the parameter:

i) $F(s) = \displaystyle\int_0^{\pi/2} \log\left(\frac{1 + s\cos x}{1 - s\cos x}\right)\frac{dx}{\cos x}$, with $|s| < 1$.

ii) $G(a) = \displaystyle\int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right) dx$, with $a \in I\!R$.

iii) $H(p) = \displaystyle\int_0^1 \frac{x^p - 1}{\log x} dx$, with $p > -1$.

iv) $I(\lambda) = \displaystyle\int_0^{\pi/2} \frac{\log(1 - \lambda^2 \sin^2 x)}{\sin x}\, dx$, with $|\lambda| < 1$.

v) $K(x) = \displaystyle\int_0^\infty \mathrm{e}^{-t^2 - x^2/t^2} dt$, with $x \in I\!R$.

Hint: *ii)* As G is an even function, it is enough to consider the case $a \geq 0$; *v)* make the change of variables $s = x/t$ to prove that $K'(x) = -2K(x)$.

Solution: i) $F(s) = \pi \arcsin s$; *ii)* $G(a) = \pi|a|$; *iii)* $H(p) = \log(p + 1)$; *iv)* $I(\lambda) = -(\arcsin\lambda)^2$; *v)* $K(x) = \sqrt{\pi}\mathrm{e}^{-2|x|}/2$.

Problem 3.9 Obtain explicitly the function

$$F(t) = \int_0^\infty \mathrm{e}^{-tx} \frac{\sin x}{x} dx\,, \qquad \forall\, t > 0\,.$$

If F is continuous at 0, deduce the value of the Dirichlet's integral

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}\,.$$

Solution: $F(t) = \pi/2 - \operatorname{arctg} t$.

Problem 3.10 Use the identity

$$\int_0^\infty \frac{\cos ax}{1+x^2}\,dx = \frac{\pi}{2}\mathrm{e}^{-a}$$

to prove that

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-\mathrm{e}^{-a}).$$

Problem 3.11 Show that

$$\int_0^{\pi/(4a)} \frac{x}{\cos^2 ax}\,dx = \frac{1}{2a^2}\Big(\frac{\pi}{2} - \log 2\Big).$$

Hint: Differentiate $\operatorname{tg} ax$ with respect to a (or integrate by parts).

Problem 3.12 Prove that

$$J(a) = \int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{\pi+2}{8a^3}\,, \qquad \text{if } a > 0.$$

Problem 3.13 Show that

$$\int_0^\pi \frac{\log(1+\cos x)}{\cos x}\,dx = \frac{\pi^2}{2},$$

computing first

$$\int_0^\pi \frac{\log(1+t\cos x)}{\cos x}\,dx.$$

Solution: The parametric integral is $\pi \cdot \arcsin t$.

Problem 3.14 Prove that

$$\int_0^\infty \frac{1-\mathrm{e}^{-x^2}}{x^2}\,dx = \sqrt{\pi}.$$

Hint: Do it similarly to the previous problem.

Problem 3.15 Let $F(\lambda) = \displaystyle\int_0^\infty \frac{dx}{x^2+\lambda}$. Write the derivatives of F and, after computing the integral, prove that for all $\lambda > 0$,

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} = \frac{1\cdot 3\cdots(2n-1)}{2^n n!}\,\frac{\pi}{2\lambda^{n+1/2}} = \frac{(2n)!\,\pi}{(n!)^2(2\sqrt{\lambda}\,)^{2n+1}}\,.$$

Solution: $F(\lambda) = \pi/(2\sqrt{\lambda}\,)$.

Problem 3.16

i) Find, for $\lambda > 0$, the integral

$$\int_0^{\pi/2} (\sin x)^\lambda \cos x \, dx.$$

ii) Use part *i*) to prove that if $\lambda > 0$ and $n \in I\!N$, we have

$$\int_0^{\pi/2} (\sin x)^\lambda \cos x \, [\log(\sin x)]^n \, dx = \frac{(-1)^n n!}{(1+\lambda)^{n+1}} \, .$$

Solution: *i*) $1/(\lambda + 1)$.

Problem 3.17 Let

$$F(x) = \int_0^{2x} \frac{\log(1+2xt)}{1+t^2} \, dt \, , \qquad x \geq 0 \, .$$

i) Check that F is differentiable on $(0, \infty)$ and show that

$$F'(x) = \frac{\log(1+4x^2)}{1+4x^2} + \frac{4x}{1+4x^2} \ \mathrm{arctg}\, 2x \, .$$

ii) Using part *i*), prove that

$$F(x) = \log \sqrt{1+4x^2} \ \mathrm{arctg}\, 2x \, .$$

Problem 3.18 Suppose that it is possible to differentiate under the integral sign, show that if a and b are positive, then

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} \, dx = \frac{\pi}{2}(b-a).$$

Problem 3.19 Suppose that it is possible to differentiate under the integral sign, show that if a and b are positive, then

$$\int_0^\infty (\mathrm{e}^{-a^2/x^2} - \mathrm{e}^{-b^2/x^2}) \, dx = \sqrt{\pi}(b-a).$$

Hint: After computing the partial derivative perform a change of variables.

Problem 3.20

i) Prove that if s is non zero, $t \in I\!R$, and if we define

$$F(t) = \int_0^\infty \mathrm{e}^{-s^2 x^2} \cos(2tx) \, dx,$$

it is verified the differential equation

$$\frac{F'(t)}{F(t)} = \frac{-2t}{s^2}.$$

ii) Obtain $F(t)$ if $F(0) = \dfrac{\sqrt{\pi}}{2|s|}$.

Solution: $F(t) = \frac{\sqrt{\pi}}{2|s|} \mathrm{e}^{-t^2/s^2}$.

Important examples of integrals depending on a parameter and its applications:

Problem 3.21 Prove the following properties of the *gamma* function

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} dt, \qquad x > 0.$$

i) Γ is continuous and differentiable. Evaluate $\Gamma^{n)}(x)$.

ii) $\Gamma(1) = \Gamma(2) = 1$; $\Gamma(1/2) = \sqrt{\pi}$.

iii) $\Gamma(x+1) = x\Gamma(x)$.

iv) Deduce from the previous result that $\lim\limits_{x \to 0^+} \Gamma(x) = +\infty$.

v) If $n \in I\!N$, $\Gamma(n+1) = n!$.

vi) Find $\Gamma(3/2)$ and $\Gamma(5/2)$.

vii) If $n \in I\!N$, $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{(2n)!}{2^{2n} n!} \sqrt{\pi}$.

viii) $\int_0^\infty \mathrm{e}^{-ax} x^n dx = n!/a^{n+1}$, if $a > 0$ and $n \in I\!N$.

Problem 3.22 If $\displaystyle\int_0^1 \log(\Gamma(x))\, dx = K$, compute the value of the integral

$$\int_\alpha^{\alpha+1} \log(\Gamma(x))\, dx$$

taking first the derivative with respect to the parameter.

Solution: $\alpha(\log \alpha - 1) + K$.

Problem 3.23 Prove the following properties of the *beta* function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \qquad p, q > 0.$$

i) $B(p,q) = B(q,p)$.

ii) B is continuous and differentiable on each variable. Prove that

$$\frac{\partial^{n+m} B}{\partial p^n \partial q^m}(p,q) = \int_0^1 x^{p-1} \big(\log x\big)^n (1-x)^{q-1} \big(\log(1-x)\big)^m dx, \qquad p, q > 0.$$

iii) If $q > 1$, then $B(p,q) = \dfrac{q-1}{p+q-1} B(p, q-1)$.

iv) If $m, n \in I\!N$,

$$(m+n+1)\, B(m+1, n+1) = \binom{m+n}{n}^{-1}.$$

v) $B(p,q) = 2\int_0^{\pi/2} (\cos t)^{2p-1} (\sin t)^{2q-1}\, dt.$

vi) $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$

vii) $B(p,q) = \displaystyle\int_0^{\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt.$

viii) $B(1/2, 1/2) = \pi$; and as a consequence, $\Gamma(1/2) = \sqrt{\pi}$.

Hint: *vi*) Use the formula $\Gamma(p) = 2\int_0^{\infty} x^{2p-1} \mathrm{e}^{-x^2} dx$ and change to polar coordinates to compute $\Gamma(p)\Gamma(q)$.

Problem 3.24 Using the *beta* and *gamma* functions, compute the following integrals

$$i) \quad \int_0^a x^2 \sqrt{a^2 - x^2}\, dx \quad (a > 0) \qquad ii) \quad \int_0^1 \sqrt{1 - t^2}\, dt$$

$$iii) \quad \int_0^1 \log^p(1/x)\, dx \quad (p > -1) \qquad iv) \quad \int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$$

$$v) \quad \int_0^{\infty} x^4 \mathrm{e}^{-7x^2}\, dx \qquad vi) \quad \int_0^{\infty} \frac{dx}{(7+x)^3 \sqrt{x}}.$$

Solution: i) $\pi a^4/16$; *ii)* $\pi/4$; *iii)* $\Gamma(p+1)$; *iv)* π; *v)* $3\sqrt{\pi}/(392\sqrt{7}\,)$.

Problem 3.25

i) Prove the formula for a, b, $c > -1$,

$$\int_D x^a y^b (1 - x - y)^c dxdy = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)}{\Gamma(a+b+c+3)},$$

where D is the triangle bounded by the line $x + y = 1$ and the coordinate axes.

ii) As an application of part *i*), prove that, for p, q, $r > 0$,

$$\int_\Omega x^{p-1} y^{q-1} z^{r-1} dxdydz = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)},$$

with Ω the tetrahedron $\Omega = \{\, x, y, z \geq 0,\ x + y + z \leq 1 \,\}$.

iii) Prove the identity

$$\int_V x^{p-1} y^{q-1} z^{r-1} dxdydz = \frac{a^p b^q c^r}{8} \frac{\Gamma(p/2)\Gamma(q/2)\Gamma(r/2)}{\Gamma((p+q+r+2)/2)},$$

where V is the interior of the ellipsoid in the first octant
$V = \left\{\, x, y, z \geq 0,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1 \,\right\}.$

iv) Compute the volume of the interior of the ellipsoid.

Hint: *i*) Apply the change of variables $u = x + y$, $v = \dfrac{y}{x+y}$, and use the relation between the *beta* and *gamma* functions. *iii*) perform a change of variables transforming ellipsoids into planes.

Solution: iv) $4\pi abc/3$.

Problem 3.26 If $f : [0,\infty) \longrightarrow \mathbb{R}$ is integrable, and has exponential growth (that is, $|f(t)| \leq c\,\mathrm{e}^{\alpha t}$, for all $t > T$, where c, α, T are certain constants depending on f), the *Laplace transform* of f is defined as

$$L(f)(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt\,.$$

i) Prove that $L(f)(s)$ converges for $s \in (\alpha, \infty)$ and that is continuous on such interval.

ii) Prove that if $|f(t)| \leq c\,\mathrm{e}^{\alpha t}$, for all $t > 0$, then

$$\big|L(f)(s)\big| \leq \frac{c}{s-\alpha}\,, \qquad s > \alpha.$$

Problem 3.27

i) Prove that if $f(t) \equiv 1$, $L(f)(s) = 1/s$, for $s > 0$.

ii) Making integration by parts, prove that if $f(t) = t^n$, $(n \in \mathbb{N})$, then

$$L(f)(s) = \frac{n!}{s^{n+1}}\,, \qquad s > 0\,.$$

iii) Using the gamma function, prove that if $f(t) = t^{-1/2}$, then

$$L(f)(s) = \sqrt{\frac{\pi}{s}}\,.$$

Does this contradict part *ii*) of the previous problem?

Problem 3.28 Prove the following properties of the Laplace transform:

i) $L(\alpha f + \beta g) = \alpha\, L(f) + \beta\, L(g), \qquad \alpha, \beta \in \mathbb{R}.$

ii) Defining $f = 0$ for $t < 0$, then if $a > 0$ we have,

$$L(f(t-a))(s) = \mathrm{e}^{-as} L(f)(s)\,.$$

iii) $L(\mathrm{e}^{-at} f(t))(s) = L(f)(s+a), \qquad a \in \mathbb{R}.$

iv) $L(f(at))(s) = \dfrac{1}{a}\, L(f(t))\Big(\dfrac{s}{a}\Big), \qquad a > 0.$

Problem 3.29 With the previous properties compute the Laplace transform of the following functions, indicating in each case its domain.

i) $f(x) = \mathrm{e}^{ax}, \quad (a \in \mathbb{R})$,

ii) $f(x) = x\,\mathrm{e}^{ax}, \quad (a \in \mathbb{R})$,

iii) $f(x) = x^n \mathrm{e}^{ax}, \quad (a \in \mathbb{R},\ n \in \mathbb{N})$,

iv) $f(x) = \sin(ax), \quad (a \in \mathbb{R})$,

v) $f(x) = \cos(ax), \quad (a \in \mathbb{R})$,

vi) $f(x) = \mathrm{e}^{-ax} \cos(bx), \quad (a, b \in \mathbb{R})$,

vii) $f(x) = \mathrm{e}^{-ax} \sin(bx), \quad (a, b \in \mathbb{R})$, *viii)* $f(x) = \sin^2 x$,

ix) $f(x) = \cos^2 x$.

Solution: *i)* $1/(s-a)$, $s > a$; *ii)* $1/(s-a)^2$, $s > a$; *iii)* $n!/(s-a)^{n+1}$, $s > a$; *iv)* $a/(s^2+a^2)$, $s > 0$; *v)* $s/(s^2+a^2)$, $s > 0$; *vi)* $(s+a)/(b^2+(s+a)^2)$, $s > -a$; *vii)* $b/(b^2+(s+a)^2)$, $s > -a$; *viii)* $2/[s(s^2+4)]$, $s > 0$; *ix)* $(s^2+2)/[s(s^2+4)]$, $s > 0$.

Problem 3.30 Let f be a continuous function on $[0,\infty)$ with exponential growth.

i) Prove that if f is differentiable on $(0,\infty)$ and f' is continuous, then

$$L(f')(s) = s\,L(f)(s) - f(0)\,.$$

ii) Deduce from *i*) that if f'' is continuous, then

$$L(f'')(s) = s^2\,L(f)(s) - s\,f(0) - f'(0)\,.$$

iii) Prove that $L(f)$ is differentiable and verifies

$$\frac{d}{ds}[L(f)(s)] = -L(t\,f(t))(s)\,.$$

iv) Prove that $L(f)$ has derivatives of all orders, verifying

$$\frac{d^n}{ds^n}[L(f)(s)] = (-1)^n L(t^n f(t))(s)\,.$$

Problem 3.31 Using the previous problem, find the Laplace transform of the following functions, showing in each case its domain:

i) $f(x) = x^n \quad (n \in I\!N)$,

ii) $f(x) = x\,\mathrm{e}^x$,

iii) $f(x) = x\cos(ax) \quad (a \in I\!R)$,

iv) $f(x) = x^2\sin(ax) \quad (a \in I\!R)$,

v) $f(x) = \sin^3 x\,$,

vi) $f(x) = \cos^3 x\,$.

Hint: *v*) $4\sin^3 x = 3\sin x - \sin 3x$; *vi*) $4\cos^3 x = 3\cos x + \cos 3x$
Solution: *i)* $n!/s^{n+1}$, $s > 0$; *ii)* $1/(s-1)^2$, $s > 1$; *iii)* $(s^2-a^2)/(s^2+a^2)^2$, $s > 0$; *iv)* $(6as^2-2a^3)/(s^2+a^2)^3$, $s > 0$; *v)* $6/[(s^2+1)(s^2+9)]$, $s > 0$; *vi)* $(s^3+7s)/[(s^2+1)(s^2+9)]$, $s > 0$.

Problem 3.32 Prove that if f is continuous on $[0,\infty)$ and has exponential growth, then the same is true for the function

$$g(x) = \int_0^x f(t)\,dt$$

and it is verified

$$L(g)(s) = \frac{1}{s}\,L(f)(s)\,.$$

Problem 3.33 Use the properties of the Laplace transform to prove that if

$$f(x) = \int_x^\infty \frac{\sin t}{t}\, dt\,,$$

then

$$L(f')(s) = \operatorname{arctg} s - \frac{\pi}{2}\,, \qquad L(f)(s) = \frac{\operatorname{arctg} s}{s}\,.$$

Problem 3.34 Evaluate the Laplace transform of the function

$$f(x) = x \int_0^x \mathrm{e}^{-at} \sin(bt)\, dt\,, \qquad a, b \in I\!R\,.$$

Solution: $b(3s^2 + 4as + a^2 + b^2)/(s^2((s+a)^2 + b^2)^2)$.

Problem 3.35

i) Express the Laplace transform of the function $f(x) = x^\alpha$, $(\alpha > -1)$, with the help of the gamma function.

ii) Find the Laplace transform of the functions

$$a)\ f(x) = \frac{\mathrm{e}^x}{\sqrt{x}}\,, \qquad b)\ f(x) = x^\alpha \mathrm{e}^{ax} \quad (a \in I\!R, \alpha > -1).$$

Solution: i) $\Gamma(\alpha+1)/s^{\alpha+1}$; *ii.a)* $\sqrt{\pi/(s-1)}$; *ii.b)* $\Gamma(\alpha+1)/(s-a)^{\alpha+1}$.

Problem 3.36 Find the function whose Laplace transform is

$$i)\ \frac{1}{s^2-1} \qquad ii)\ \frac{1}{(s+1)^2}$$

$$iii)\ \frac{1}{s(s+1)^2} \qquad iv)\ \frac{1}{s^n} \quad (n \in I\!N)$$

$$v)\ \frac{1}{(s-1)^2(s^2+1)} \qquad vi)\ \frac{4s+12}{s^2+8s+16}$$

$$vii)\ \frac{s\,\mathrm{e}^{-\pi s/2}}{s^2+a^2} \qquad viii)\ \frac{1}{\sqrt{s}}.$$

Solution: i) $\sinh x = (\mathrm{e}^x - \mathrm{e}^{-x})/2$; *ii)* $x\,\mathrm{e}^{-x}$; *iii)* $1 - (x+1)\,\mathrm{e}^{-x}$; *iv)* $x^{n-1}/(n-1)!$; *v)* $\frac{1}{2}((x-1)\mathrm{e}^x + \cos x)$; *vi)* $4(1-x)\mathrm{e}^{-4x}$; *vii)* $\cos(a(x-\pi/2))$ if $x \geq \pi/2$, 0 if $x < \pi/2$; *viii)* $1/\sqrt{\pi x}$.

Problem 3.37 Let $f : (0,\infty) \longrightarrow I\!R$ be a piecewise continuous function with exponential growth.

i) Prove that if f is periodic of period P, that is, $f(x+P) = f(x)$ for all $x > 0$, then

$$L(f)(s) = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt\,.$$

ii) As an application of the previous formula, compute the transform of Laplace of the function $f(x) = x - [x]$, where $[x]$ denotes the integer part of x.

Hint: *i)* Divide the integral that defines $L(f)$ in two parts, one defined on $[0, P]$ and the other on $[P, \infty]$. Make an appropriate change of variables in the second integral to exploit the periodicity of f.

Solution: ii) $\frac{1}{s}(\frac{1}{s} - \frac{1}{e^s - 1})$.

Problem 3.38 Solve the following initial value problems

$$i) \quad \begin{cases} y' - 3y = e^{2t} \\ y(0) = 1, \end{cases} \qquad ii) \quad \begin{cases} y' + 3y = \sin 2t \\ y(0) = 0 \end{cases}$$

$$iii) \quad \begin{cases} y' - 5y = \cos 3t \\ y(0) = 1/2 \end{cases} \qquad iv) \quad \begin{cases} y'' - y = e^{2t} \\ y(0) = 0, \ y'(0) = 1 \end{cases}$$

$$v) \quad \begin{cases} y'' + 16y = \cos 4t \\ y(0) = 0, \ y'(0) = 1 \end{cases} \qquad vi) \quad \begin{cases} y'' + 2y' + y = e^{-3t} \\ y(0) = 1, \ y'(0) = 0 \end{cases}$$

$$vii) \quad \begin{cases} y'' - 6y' + 9y = t^2 e^{3t} \\ y(0) = 2, \ y'(0) = 6 \end{cases} \qquad viii) \quad \begin{cases} y'' + 4y' + 6y = 1 + e^{-t} \\ y(0) = 0, \ y'(0) = 0. \end{cases}$$

Solution: i) $y(t) = 2e^{3t} - e^{2t}$; *ii)* $y(t) = (2e^{-3t} - 2\cos 2t + 3\sin 2t)/13$;
iii) $y(t) = (22e^{5t} - 5\cos 3t + 3\sin 3t)/34$; *iv)* $y(t) = (e^{2t} - e^{-t})/3$;
v) $y(t) = [(2 + t)\sin 4t]/8$; *vi)* $y(t) = (e^{-3t} + 3e^{-t} + 6te^{-t})/4$;
vii) $y(t) = (24 + t^4)e^{3t}/12$; *viii)*; $y(t) = (1 + 2e^{-t} - 3e^{-2t}\cos(\sqrt{2}\,t) - 2\sqrt{2}\,e^{-2t}\sin(\sqrt{2}\,t))/6$.

Problem 3.39 Let f and g be continuous functions on $([0, \infty)$, such that $f(x) = g(x) = 0$ for all $x < 0$, we define the *convolution* of f and g as

$$(f * g)(x) = \int_0^\infty f(y)\, g(x - y)\, dy = \int_0^x f(y)\, g(x - y)\, dy.$$

i) Show that the convolution is commutative, that is, $f * g = g * f$.

ii) Prove that if f or g are differentiable (even though the other is not differentiable), then $f * g$ is differentiable, compute $(f * g)'$.

iii) Show that if f and g have exponential growth, then its convolution has it also and the following identity is verified

$$L(f * g) = L(f)\, L(g)\,.$$

iv) Find $L(f * g)$ if$f(x) = e^x, \ g(x) = \sin x$, and $x \geq 0, \ (= 0$ if $x < 0)$.

v) Find the function whose Laplace transform is

$$a) \ \frac{1}{(s-1)(s-4)} \qquad b) \ \frac{1}{(s^2 + a^2)^2}.$$

vi) Find the function $f(x)$ verifying the identity

$$f(x) + \int_0^x f(y)\, e^{x-y}\, dy - 3x^2 + e^{-x} = 0.$$

Solution: ii) $(f * g)' = f * (g') = (f') * g$; *iv)* $1/[(s - 1)(s^2 + 1)]$; *v.a)* $\frac{1}{3}(e^{4t} - e^t)$;
v.b) $\frac{1}{2a^3}(\sin at - at\cos at)$; *vi)* $f(x) = 3x^2 - x^3 + 1 - 2e^{-x}$.

4 LINE AND PATH INTEGRALS

Parametrizations of important curves:

Circumference:	$(x-a)^2 + (y-b)^2 = r^2$	$\Rightarrow$	$\gamma_1(t) = (a + \cos t, b + \sin t)$.
Ellipse:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\Rightarrow$	$\gamma_2(t) = (a \cos t, b \sin t)$.
Helix:		$\Rightarrow$	$\gamma_3(t) = (\cos t, \sin t, t)$.

Problem 4.1 Sketch the previous curves.

Problem 4.2 Integrate

i) $f(x,y) = 2xy^2$ over the first quadrant of the circumference of radius R.

ii) $f(x,y,z) = (x^2+y^2+z^2)^2$ along the arc of the circular helix $\mathbf{r}(t) = (\cos t, \sin t, 3t)$, from $(1,0,0)$ to $(1,0,6\pi)$.

Solution: i) $2R^4/3$; *ii)* $2\pi\sqrt{10}(5 + 120\pi^2 + 1296\pi^4)/5$.

Problem 4.3 Determine the length and the mass of a thread whose shape is the parabola $y = x^2$ from $(0,0)$ to $(2,4)$ and whose density is $\rho(x,y) = x$.

Solution: The length is $\sqrt{17} + (\log(4+\sqrt{17}))/4$ and the mass is $(17^{3/2}-1)/12$.

Problem 4.4 Evaluate the following integrals, if the closed curves have positive orientation, that is, counterclockwise:

i) $\displaystyle\int_g (x-y)dx + (x+y)dy$, where g is the line segment joining (1,0) to (0,2).

ii) $\displaystyle\int_C x^3dy - y^3dx$, where C is the circumference $\{x^2+y^2=1\}$.

iii) $\displaystyle\int_\Gamma \frac{dx+dy}{|x|+|y|}$, where Γ is the square of vertices (1,0), (0,1), $(-1,0)$ and $(0,-1)$.

iv) $\displaystyle\int_\rho (x+2y)dx + (3x-y)dy$ where ρ is the ellipse $x^2+4y^2=4$.

v) $\displaystyle\int_R \frac{y^3dx - xy^2dy}{x^5}$, where R is the curve $x = \sqrt{1-t^2}$, $y = t\sqrt{1-t^2}$, $-1 \le t \le 1$.

Solution: i) 7/2; *ii)* $3\pi/2$; *iii)* 0; *iv)* 2π; *v)* $-\pi/2$.

Problem 4.5 Evaluate:

i) $\displaystyle\int_\gamma y\,dx - x\,dy + z\,dz$, where γ is the intersection curve of the cylinder $x^2+y^2=a^2$ with the plane $z - y = a$, oriented counterclockwise.

ii) $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (2xy+z^2, x^2, 2xz)$, where γ is the intersection of the plane $x = y$ with the sphere $x^2+y^2+z^2=a^2$, positively oriented.

iii) $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y,z,x)$, where γ is the curve intersection of $x^2+y^2=2x$ and $x=z$.

Solution: *i)* $-2\pi a^2$; *ii)* 0; *iii)* 0.

Problem 4.6 Find the value of b that minimizes the work done in moving a particle by the force field $\mathbf{F}(x,y) = (3y^2+2, 16x)$, from $(-1,0)$ to $(1,0)$, along the semiellipse $b^2x^2+y^2=b^2$, $y \geq 0$.

Solution: The work done is $W(b) = 4b^2 - 8\pi b + 4$ and the minimum work is $4-4\pi^2$, obtained for $b=\pi$.

Problem 4.7 Consider the force field $\mathbf{F}(x,y) = (cxy, x^6y^2)$, $a,b,c>0$. Find the parameter a in terms of c such that the work done in moving a particle along the parabola $y=ax^b$ from $x=0$ to $x=1$ will not depend on b.

Solution: the work is $\frac{3ac+a^3b}{3(b+2)}$, hence, $a=0$ or $a=\sqrt{3c/2}$.

Problem 4.8 Evaluate the work done in moving a particle under a force field (given in polar coordinates) $\mathbf{F}(r,\theta) = (-4\sin\theta, 4\sin\theta)$, along the path $r=\mathrm{e}^{-\theta}$ from (1,0) to the origin.

Solution: 8/5.

Problem 4.9 Let $\mathbf{F}(x,y,z) = (\sin y + z, x\cos y + \mathrm{e}^z, x + y\mathrm{e}^z)$.

i) Prove that the integral over any piecewise C^1 simple closed curve is equal to 0.

ii) Obtain a potential of F, that is, find ϕ such that $\mathbf{F} = \nabla\phi$.

Solution: *ii)* $\phi(x,y,z) = x(\sin y + z) + y\mathrm{e}^z$.

Problem 4.10 Evaluate $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (2xz\mathrm{e}^{x^2+y^2}, 2yz\mathrm{e}^{x^2+y^2}, \mathrm{e}^{x^2+y^2})$ and γ the path on $\mathbb{R}^3$ given by $\mathbf{r}(t) = (t,t^2,t^3)$, $0\leq t\leq 1$.
Hint: Prove that $\mathbf{F}$ is a gradient field.

Solution: e^2.

Problem 4.11 Given the curve on $\mathbb{R}^3$, $\gamma(t) = \Big(\mathrm{e}^{t^2} + t(1-\mathrm{e}) - 1, \sin^5(\pi t), \cos(t^2-t)\Big)$, $t \in [0,1]$, and the vector field

$$\mathbf{F}(x,y,z) = (y+z+x^4\sin x^5,\ x+z+\operatorname{arctg} y,\ x+y+\sin^2 z).$$

i) Find $\displaystyle\int_\gamma \mathbf{F}$.

ii) Does it exist f such that $\nabla f = \mathbf{F}$? If this is the case, find f.

Solution: *i)* 0; *ii)* $f(x,y,z) = xy+xz+yz-\frac{1}{5}\cos x^5 + y\operatorname{arctg} y - \frac{1}{2}\log(1+y^2) + \frac{z}{2} - \frac{1}{4}\sin 2z$.

Problem 4.12 Given the curve on $I\!R^3$, $\Gamma = \{\, x^2 + y^2 = 1,\ z = y^2 - x^2 \,\}$, positively oriented, and the vector field $\mathbf{F}(x, y, z) = (y^3, \mathrm{e}^y, z)$.

i) Find $\displaystyle\int_\Gamma \mathbf{F}$.

ii) Does it exist f such that $\nabla f = \mathbf{F}$?

Solution: $i)$ $-\,3\pi/4$; $ii)$ No.

Problem 4.13 Determine a and b such that the vector field

$$\mathbf{w}(x, y) = \mathrm{e}^{2x+3y}\Big(a \sin x + a \cos y + \cos x,\ b \sin x + b \cos y - \sin y\Big)$$

is irrotational (that is, its curl is 0) and find its potential.

Solution: $a = 2,\ b = 3;\ \varphi(x, y) = \mathrm{e}^{2x+3y}(\sin x + \cos y) + C$.

Problem 4.14 Consider the vector field

$$\mathbf{F}(x, y) = \Big(\frac{\log x + \log y}{x}\ ,\ \frac{\log x + \log y}{y}\Big),$$

defined on the domain $D = \{(x, y) :\ x > 0\,,\ y > 0\}$.

i) Evaluate $\displaystyle\int_\gamma \mathbf{F}$, where γ is the arc of the hyperbola $xy = a$ $(a > 0)$, such that $x_1 \le x \le x_2$.

ii) Let A be any point of the hyperbola $xy = a$ $(a > 0)$, B any point of the hyperbola $xy = b$ $(b > a)$, and γ any C^1 path, contained on D joining A to B, prove that

$$\int_\gamma \mathbf{F} = \frac{1}{2}\ \log\frac{b}{a}\ \log(ab)\,.$$

Hint: $\mathbf{F}$ is conservative.

Solution: $i)$ 0.

Problem 4.15 Evaluate $\displaystyle\int_\gamma (5 - xy - y^2)dx - (2xy - x^2)dy$, where γ is the square of vertices (0,0), (1,0), (1,1) and (0,1), compute it directly and also applying Green's Theorem.

Solution: 3/2.

Problem 4.16 Let f be a C^1 function on $I\!R$. Let

$$P(x, y) = \mathrm{e}^{x^2} - \frac{y}{3 + \mathrm{e}^{xy}}, \quad Q(x, y) = f(y),$$

and γ the boundary of the square $[0, 1] \times [0, 1]$ oriented in the positive direction. Evaluate $\displaystyle\int_\gamma P dx + Q dy$.

Solution: $(1 - \log(\mathrm{e} + 3) + \log 4)/3$.

Problem 4.17 Evaluate $\int_\Gamma xy\,dx + \sin^2(\mathrm{e}^{\cos y})\,dy$, where Γ is the curve $y = \mathrm{e}^{-x^2}$, for $x \in (-\infty, \infty)$.
Hint: Apply Green's Formula to the same integral over the curve Γ_R, formed by the line segment $(-R, R)$, the function $y = \mathrm{e}^{-x^2}$ on the same interval and the vertical line segments joining both of them, positively oriented ; after, take the limit when $R \to \infty$.

Solution: 0.

Problem 4.18 Let the functions $P(x, y) = y/(x^2 + y^2)$ and $Q(x, y) = -x/(x^2 + y^2)$. Let C be a piecewise C^1 closed curve, defined outside the origin, such that is the boundary of a region D.

i) Prove that $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ for $(x, y) \neq (0, 0)$.

ii) If $(0, 0) \in D$, prove that $\int_C P\,dx + Q\,dy = \pm 2\pi$.

iii) If $(0, 0) \notin D$, compute $\int_C P\,dx + Q\,dy$.

Solution: iii) 0.

Problem 4.19 Evaluate $\int_\gamma \dfrac{-y\,dx + (x-1)\,dy}{(x-1)^2 + y^2}$, where γ is a piecewise C^1 simple closed curve, containing (1,0) in its interior, oriented in the positive direction.

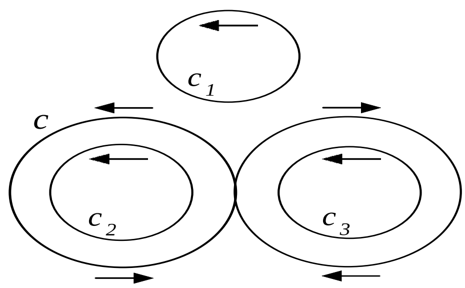
Solution: 2π.

Problem 4.20 Let $P, Q \in C^1(I\!R^2)$ be two scalar fields such that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ everywhere on the plane but at three points. Let C_1, C_2 and C_3 be three disjoint circles surrounding them and $I_k = \displaystyle\int_{C_k} P\,dx + Q\,dy$. If $I_1 = 12$, $I_2 = 10$ and $I_3 = 15$,

i) compute $\displaystyle\int_C P\,dx + Q\,dy$, where C is the curve of the figure, surrounding C_2 and C_3;

ii) sketch γ, such that $\displaystyle\int_\gamma P\,dx + Q\,dy = 1$;

iii) if $I_1 = 12$, $I_2 = 9$ and $I_3 = 15$, prove that is impossible to find such a curve γ.



Hint: *iii*) the integral is proportional to 3.
Solution: i) -5; *ii*) γ is any curve surrounding once C_2 in the positive direction, once C_3 in the positive direction and twice C_1 in the negative direction.

Problem 4.21

i) Let A be the area of a region D, bounded by C, a piecewise C^1 simple closed curve. Prove that

$$A = \frac{1}{2}\int_C -y\,dx + x\,dy = \int_C x\,dy = -\int_C y\,dx\,,$$

and prove also that in polar coordinates it takes the form

$$A = \frac{1}{2}\int_C r^2(\theta)\,d\theta.$$

ii) Evaluate the area of the interior of the loop of the curve parametrized as $\mathbf{s}(t) = (t^2 - 1, t^3 - t)$.

iii) Evaluate the area of the cardioid, given in polar coordinates as $r(\theta) = a(1 - \cos\theta)$, $(0 \le \theta \le 2\pi)$.

Solution: ii) $8/15$; *iii*) $3\pi a^2/2$.

Problem 4.22

i) Evaluate $\displaystyle\int_D (x + 2y)dxdy$, where D is the cycloid's arc $x = t - \sin t,\ y = 1 - \cos t$, $0 \le t \le 2\pi$.

ii) Evaluate $\displaystyle\int_D xy^2 dxdy$, where D is the region bounded by the astroid $x = \cos^3 t,\ y = \sin^3 t$, $0 \le t \le \pi/2$ and the coordinate axes.

iii) Evaluate $\int_D y^2 dxdy$, where D is the region bounded by the curve $x = a(t - \sin^2 t),\ y = a\sin^2 t$, $0 \le t \le \pi$, and the line joining its endpoints.

Solution: i) $\pi(3\pi + 5)$; *ii)* $8/2145$; *iii)* $5\pi a^4/48$.

Problem 4.23 Let $a, b > 0$.

i) Prove (by integrating) that:

$$\int_0^{\pi/2} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{\pi}{2ab} \,.$$

ii) Using part *i)* and taking derivatives inside the integral, prove that

$$\int_0^{\pi/2} \frac{\sin^2 t}{(a^2 \cos^2 t + b^2 \sin^2 t)^2}\, dt = \frac{\pi}{4ab^3} \,.$$

iii) Using part *ii)*, prove that

$$\int_\gamma \frac{y^3\, dx - xy^2\, dy}{(x^2 + y^2)^2} = \pi \,,$$

where γ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ clockwise oriented.

Problem 4.24 Let $r = \|\mathbf{x}\|$, defined for all $\mathbf{x} \in I\!R^n$, .

i) Find ∇f for $f(\mathbf{x}) = r^\alpha$.

ii) Do the same for $f(\mathbf{x}) = g(r)$, where g is a differentiable function of one variable.

iii) Find div $\mathbf{F}$ for $\mathbf{F}(\mathbf{x}) = r^\alpha \mathbf{x}$.

iv) Find $\Delta f = \text{div}\,(\nabla f)$ for $f(\mathbf{x}) = r^\alpha$.

v) Find a potential for the force field $\mathbf{F}(\mathbf{x}) = g(r)\mathbf{x}$, where g is a continuous function of one variable.

Solution: i) $\alpha r^{\alpha-2}\mathbf{x}$; *ii)* $g'(r)\mathbf{x}/r$; *iii)* $(\alpha + n)r^\alpha$; *iv)* $\alpha(\alpha + n - 2)r^{\alpha-2}$; *v)* $\varphi(r) = \int_0^r sg(s)ds$.

Problem 4.25 Let D be a region defined in $I\!R^2$ bounded by the regular closed curve C, and let $u,\ v \in C^2(\overline{D})$. If $\mathbf{n}$ denotes the unit normal vector exterior to the curve, use Divergence's Theorem to prove the following identities:

$$\begin{aligned}
&i) & \int_C \frac{\partial u}{\partial \mathbf{n}}\, ds &= \int_D \Delta u\, dxdy \\
&ii) & \int_C v\frac{\partial u}{\partial \mathbf{n}}\, ds &= \int_D (v\Delta u + \nabla u \cdot \nabla v)\, dxdy \\
&iii) & \int_C \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\right) ds &= \int_D (v\Delta u - u\Delta v)\, dxdy \,.
\end{aligned}$$

Problem 4.26 Let u and v be two C^1 class scalar fields defined on an open containing the unit disk D. If $\mathbf{F}(x,y) = (v(x,y), u(x,y))$ and $\mathbf{G}(x,y) = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)$. Evaluate

$$\int_D \mathbf{F} \cdot \mathbf{G}\, dxdy\,,$$

where u and v verifies that $u = 1$ and $v = y$ at the unit circumference.

Solution: $-\pi$.

5 SURFACE INTEGRALS

Parametrizations of important surfaces:

Sphere:	$x^2+y^2+z^2 = R^2$	$\Rightarrow$	$\phi_1(\theta,\varphi) = (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$.
Cylinder:	$x^2+y^2 = R^2$	$\Rightarrow$	$\phi_2(\theta,z) = (R\cos\theta, R\sin\theta, z)$.
Cone:	$\sqrt{x^2+y^2} = z$	$\Rightarrow$	$\phi_3(r,\theta) = (r\cos\theta, r\sin\theta, r)$.
Paraboloid:	$x^2+y^2 = z$	$\Rightarrow$	$\phi_4(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.
Helicoid:		$\Rightarrow$	$\phi_5(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$

Problem 5.1 Sketch the graphs of the previous surfaces.

Problem 5.2 Evaluate the area of the following surfaces:

i) sphere of radius R;

ii) circular cone parametrized by $\mathbf{r}(u,v) = (u\cos v, u\sin v, u)$, where $0 \le u \le a$ and $0 \le v \le 2\pi$.

iii) portion of the paraboloid $z = x^2+y^2$ inside the cylinder $x^2+y^2 = a^2$;

iv) portion of the cylinder $x^2+z^2 = 16$ bounded by the cylinder $x^2+y^2 = 16$.

Solution: i) $4\pi R^2$; *ii)* $\pi a^2\sqrt{2}$; *iii)* $\pi((1+4a^2)^{3/2}-1)/6$; *iv)* 128.

Problem 5.3

i) Deduce the area's formula of the revolution surface obtained by rotating the graph $y = f(x)$, $0 < a \le x \le b$, about the vertical axis:

$$A = 2\pi \int_a^b x\sqrt{1+(f'(x))^2}\, dx\,,$$

using the parametrization $\mathbf{s}(r,\theta) = (r\cos\theta, r\sin\theta, f(r))$, where $a \le r \le b$ and $0 \le \theta \le 2\pi$.

ii) Find the surface area of the torus obtained by rotating about the vertical axis the graph $(x-R)^2+y^2 = c^2$, $0 < c < R$.

iii) Deduce the corresponding parametrization to obtain the analogous formula in the case of rotating the graph $y = f(x)$, $a \le x \le b$, about the horizontal axis.

Solution: ii) $4\pi^2 Rc$; *iii)* $\mathbf{s}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta))$.

Problem 5.4 Let $W = \{1 \leq z \leq (x^2+y^2)^{-1/2}\}$ be a region of $I\!R^3$. Prove that W has finite volume but its boundary has infinite area.

Solution: $V = \pi$.

Problem 5.5 Find the moment of inertia with respect to a diameter of an homogeneous spherical shell of mass m and radius a.

Solution: $2ma^2/3$.

Problem 5.6 Evaluate $\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ ($\mathbf{n}$ the outward normal vector) in the following cases:

i) $\mathbf{F}(x,y,z) = (x^2, y^2, z^2)$ and S the boundary of the cube $0 \leq x,\ y,\ z \leq 1$.

ii) $\mathbf{F}(x,y,z) = (xy, -x^2, x+z)$ and S the portion of the plane $2x+2y+z=6$ on the first octant, with $\mathbf{n}$ the normal vector with positive third component.

iii) $\mathbf{F}(x,y,z) = (xz^2, x^2y - z^2, 2xy + y^2z)$ and S the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

iv) $\mathbf{F}(x,y,z) = (2x^2 + \cos yz, 3y^2z^2 + \cos(x^2+z^2), e^{y^2} - 2yz^3)$ and S the surface of the solid generated by intersecting the cone $z \geq \sqrt{x^2+y^2}$ with the ball $x^2+y^2+z^2 \leq 1$.

Solution: *i)* 3; *ii)* 27/4; *iii)* $2a^5\pi/5$; *iv)* 0.

Problem 5.7
Consider the surface

$$S = \{\, (x,y,z) \in I\!R^3 \ :\ x^2+y^2+z^2 = 1,\ y \geq 0 \,\}$$

(oriented with the normal vector exterior to the unit sphere) and the function

$$\mathbf{F}(x,y,z) = (x+z, y+z, 2z).$$

i) Evaluate $\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS$.

ii) Evaluate $\int_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Solution: *i)* π; *ii)* $8\pi/3$.

Problem 5.8 Using Stokes's Theorem compute the integral $\displaystyle\int_S \text{curl } \mathbf{F}$ in the following cases, where S is oriented with outward normal vector:

i) $\mathbf{F}(x,y,z) = (x^2y^2, yz, xy)$ and S the paraboloid $z = a^2 - x^2 - y^2,\ z \geq 0$.

ii) $\mathbf{F}(x,y,z) = ((1-z)y, ze^x, x\sin z)$ and S the upper hemisphere of radius a.

iii) $\mathbf{F}(x,y,z) = (x^3+z^3, e^{x+y+z}, x^3+y^3)$ and $S = \{\, x^2+y^2+z^2 = 1,\ y \geq 0 \,\}$.

Solution: *i)* 0; *ii)* $-\pi a^2$; *iii)* 0.

Problem 5.9 Consider the vector field

$$\mathbf{F}(x,y,z) = \left(y\,,\ x^2\,,\ (x^2+y^4)^{3/2}\,\sin(e^{\sqrt{xyz}}\,)\right).$$

Evaluate $\displaystyle\int_S \text{curl}\mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ denotes the inward normal vector to the semiellipsoid

$$S = \{(x,y,z) :\ 4\,x^2 + 9\,y^2 + 36\,z^2 = 36\,,\ z \geq 0\}\,.$$

Solution: 6π.

Problem 5.10 Let $\mathbf{F}(x, y, z) = (2y, 3z, x)$ and T be the triangle with vertices $A(0, 0, 0)$, $B(0, 2, 0)$ and $C(1, 1, 1)$.

i) Find an orientation of the triangle's surface and the one induced at the boundary.

ii) Evaluate the line integral of the vector field $\mathbf{F}$ over the boundary of T.

Solution: *i)* $\mathbf{n} = (1, 0, -1)$; the orientation of the boundary is $A \to B \to C \to A$; *ii)* -1.

Problem 5.11 Redo parts *i)*, *iii)* and *iv)* of problem 5.6, using the vector calculus theorems.

Problem 5.12 Consider the function

$$\mathbf{F}(x, y, z) = (y \sin(x^2 + y^2), -x \sin(x^2 + y^2), z(3 - 2y))$$

and the region

$$W = \{(x, y, z) \in I\!R^3 \,/\, x^2 + y^2 + z^2 \leq 1, z \geq 0\}.$$

Evaluate $\displaystyle\int_{\partial W} \mathbf{F}$, where ∂W denotes the boundary of W.

Solution: 2π.

Problem 5.13 Verify Stokes's Theorem for:

i) $\mathbf{F}(x, y, z) = (y^2, xy, xz)$, on the paraboloid $z = a^2 - x^2 - y^2,\ z \geq 0$.

ii) $\mathbf{F}(x, y, z) = (-y^3, x^3, z^3)$ on $S = \{ z = y,\ y \geq 0,\ x^2 + y^2 \leq 1 \}$.

Solution: *i)* 0; *ii)* $3\pi/4$.

Problem 5.14 Evaluate the integral $\displaystyle\int_S \mathbf{F}$, where

i) $\mathbf{F}(x, y, z) = (18z, -12, 3y)$, and S is the region of the plane $2x + 3y + 6z = 12$ on the first octant.

ii) $\mathbf{F}(x, y, z) = (x^3, x^2y, x^2z)$, and S is the closed surface formed by the cylinder $x^2 + y^2 = a^2,\ 0 \leq z \leq b$, and its upper and lower bases.

iii) $\mathbf{F}(x, y, z) = (4xz, -y^2, yz)$, and S is the boundary surface of the cube $0 \leq x,\ y,\ z \leq 1$.

iv) $\mathbf{F}(x, y, z) = (x, y, z)$, and S is a simple closed surface.

Solution: *i)* 24; *ii)* $5\pi a^4 b/4$; *iii)* $3/2$; *iv)* $3|\Omega|$, where $S = \partial\Omega$.

Problem 5.15 Compute the flux of the vector field $\mathbf{F}(x, y, z) = (y^2, yz, xz)$ over the surface of the tetrahedron bounded by $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$, oriented with outward normal.

Solution: $1/12$.

Problem 5.16 Suppose a temperature function on each point of the space is proportional to the square of the vertical axis distance. Consider the region $V = \{ (x, y, z) \in I\!R^3 \ :\ x^2 + y^2 \leq 2z,\ z \leq 2 \}$.

i) Compute the volume of V.

ii) Compute the average temperature on V.

iii) Compute the gradient flux of the temperature out of ∂V.

Solution: *i)* 4π; *ii)* $4\alpha/3$ (if α is the constant of proportionality); *iii)* $16\alpha\pi$.

Problem 5.17 Let D be a region on the plane with boundary $C = \partial D$. Suppose that the inertia moments of D with respect to the coordinate axes are $I_x = a$ and $I_y = b$ respectively. If the density is 1, $\mathbf{n}$ is the outward unit normal vector to C, and $r = \|\mathbf{x}\| = \sqrt{x^2 + y^2}$, compute the integral

$$\int_C \nabla r^4 \cdot \mathbf{n}\, ds.$$

Solution: $16(a + b)$.

Problem 5.18 Let S be the unit sphere on $\mathbb{R}^3$, and let φ be a nonzero function verifying $\|\nabla\varphi\|^2 = 4\varphi$ and $\operatorname{div}(\varphi\nabla\varphi) = 10\varphi$. Evaluate

$$\int_S \frac{\partial\varphi}{\partial\mathbf{n}}\, dS,$$

where $\mathbf{n}$ is the outward unit normal vector to S.
Hint: From the two equations involving φ, compute $\Delta\varphi$.

Solution: 8π.

Problem 5.19 Let S be the sphere of radius a oriented with its outward normal vector, and let the vector field $\mathbf{F}(x, y, z) = (\sin yz + \mathrm{e}^z, x\cos z + \log(1 + x^2 + z^2), \mathrm{e}^{x^2+y^2+z^2})$. Evaluate $\displaystyle\int_S \mathbf{F}$.

Solution: 0.

Problem 5.20 Let $S = S_1 \cup S_2$, where S_1 and S_2 are the surfaces

$$S_1 = \{\, x^2 + y^2 = 1,\ 0 \le z \le 1 \,\}, \qquad S_2 = \{\, x^2 + y^2 + (z-1)^2 = 1,\ z \ge 1 \,\},$$

and let the vector field $\mathbf{F}(x, y, z) = (zx + z^2y + x, z^3yx + y, z^4x^2)$.

i) Evaluate $\displaystyle\int_S \operatorname{curl}\mathbf{F}$ using Stokes's Theorem.

ii) Evaluate the same integral using Divergence's Theorem.

Solution: 0.

Problem 5.21 Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Find $\displaystyle\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ is the inward unit normal vector to $\partial\Omega$, and

$$\mathbf{F}(x, y, z) = \left(\mathrm{e}^{y^2+z^2} + \int_0^x \frac{\mathrm{e}^{t^2+y^2}}{\sqrt{t^2+y^2}}\, dt,\ \sin(x^2 + \mathrm{e}^z),\ h(x, y)\right),$$

$$\Omega = \{\, (x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \le 1,\ 0 \le z \le \sqrt{x^2+y^2},\ x \ge 0,\ y \ge 0 \,\}.$$

Solution: $\pi(1-\mathrm{e})/4$.

Problem 5.22 Consider the vector field

$$\mathbf{F}(x,y,z) = \Big(y\,\mathrm{e}^z\,,\ \int_0^x \mathrm{e}^{-t^2+\cos z}dt\,,\ z(x^2+y^2)\Big)\,.$$

Evaluate $\displaystyle\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}$, where $\mathbf{n}$ denotes the outward normal vector to the boundary of the region

$$\Omega = \{(x,y,z):\ x^2+y^2+z^2 < a^2\,,\ x^2+y^2 < z^2\}\,.$$

Solution: $(8-5\sqrt{2})\pi a^5/15$.

Problem 5.23 A vector field on $I\!R^3$ has the form $\mathbf{F}(x,y,z) = (P_1(x,y)+P_2(x,z), x+Q(y,z), R(x,y,z))$, with $P_1, P_2, Q, R \in \mathcal{C}^2(I\!R^3)$. If Γ_h is the boundary of the cylinder's section $x^2+y^2=1$ at height h, prove that $\displaystyle\int_{\Gamma_h} \mathbf{F}\cdot d\mathbf{r}$ is independent of h.