Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

Primer Curso de INGENIERÍA DE TELECOMUNICACIÓN. **CALCULO II**.
Examen 15 de Septiembre de 2008.

Apellidos .... Nombre ....

D.N.I. .... Grupo ....

EXPLAIN CAREFULLY EACH PROBLEM. **Time length: 3.30 hours.**

PROBLEM 1. (3 p.) Given the following improper integral depending on a parameter

$$\Phi(\alpha, n) = \int_0^\infty \frac{dx}{(x^2+\alpha)^n}, \quad \alpha > 1,$$

(a) Prove that
$$\Phi(\alpha, 1) = \int_0^\infty \frac{dx}{x^2+\alpha}, \quad \alpha > 1,$$
is a convergent improper integral.

(b) Show that
$$\Phi(\alpha, 1) = \frac{\pi}{2\sqrt{\alpha}}.$$

(c) Evaluate
$$\Phi(\alpha, 2) = \int_0^\infty \frac{dx}{(x^2+\alpha)^2} \qquad \text{and} \qquad \Phi(1, 2).$$

(d) Prove that
$$\Phi(1, n+1) = \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} = \int_0^{\pi/2} \cos^{2n}\theta d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2}.$$

Solution:

(a) Using Weierstrass' proof, as
$$\frac{1}{x^2+\alpha} \le \frac{1}{x^2+1} \quad \text{with} \quad \alpha \ge 1, \quad \text{and} \quad \int_0^\infty \frac{dx}{x^2+1}$$
converges, we have that
$$\int_0^\infty \frac{dx}{x^2+\alpha}$$
is also a convergent integral.

(b)
$$\Phi(\alpha, 1) = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+\alpha} = \lim_{b\to\infty}\frac{1}{\sqrt{\alpha}}\arctan\frac{b}{\sqrt{\alpha}} = \frac{\pi}{2\sqrt{\alpha}}$$

(c) From (b),
$$\Phi(\alpha, 1) = \int_0^\infty \frac{dx}{x^2+\alpha} = \frac{\pi}{2\sqrt{\alpha}}.$$
Observe that for each $\alpha \in [1, \infty)$,
$$f(x, \alpha) = \frac{1}{x^2+\alpha},$$

is a continuous function on $[0, \infty)$, with continuous partial derivatives with respect to the variable $\alpha \in [1, \infty)$, for each $x \in [0, \infty)$. Note also that the improper integral

$$\int_0^\infty \frac{dx}{(x^2+\alpha)^2} dx,$$

is convergent, because of

$$\frac{1}{(x^2+\alpha)^2} \leq \frac{1}{(x^2+1)^2} \quad \text{and} \quad \int_0^\infty \frac{dx}{(x^2+1)^2} dx \quad \text{converges.}$$

Then, differentiating both sides with respect to α, we obtain that

$$\frac{\partial \Phi(\alpha, 1)}{\partial \alpha} = \int_0^\infty \frac{\partial \alpha}{\partial \alpha} \left(\frac{1}{x^2+\alpha} \right) dx = -\int_0^\infty \frac{dx}{(x^2+\alpha)^2} dx = -\frac{\pi}{4} \alpha^{-\frac{3}{2}}.$$

Taking the limit $\alpha \to 1+$, we find that

$$\Phi(1, 2) = \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

(d) Differentiating n times both sides of the equation

$$\Phi(\alpha, 1) = \int_0^\infty \frac{dx}{x^2+\alpha} = \frac{\pi}{2\sqrt{\alpha}},$$

we have that

$$(-1)(-2)\cdots(-n) \int_0^\infty \frac{dx}{(x^2+\alpha)^{n+1}} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\frac{\pi}{2}\alpha^{-(2n+1)/2}.$$

Letting $\alpha \to 1+$, we have that

$$\int_0^\infty \frac{dx}{(x^2+\alpha)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}.$$

Substituting $x = \tan\theta$, we obtain, finally, $\int_0^{\pi/2} \cos^{2n}\theta d\theta$.

PROBLEM 2. (3 p.) Given the following vector field

$$\vec{F}(x, y) = (P, Q) = (2x^3 - y^3, x^3 + y^3).$$

(a) (1 p.) Determine if it is a conservative field or not. If it is conservative, then find the potential function.

(b) (2 p.) Verify Green's Theorem for such vector field $\vec{F}$, on the disk of radius R: $\{x^2 + y^2 \leq R^2\}$.

Solution:

(a) $\dfrac{\partial}{\partial y}(2x^3 - y^3) = -3y^2$ and $\dfrac{\partial}{\partial x}(x^3 + y^3) = 3x^2$. As the mixed derivatives are not equal, we conclude that the vector field is not conservative.

(b) Applying Green's theorem, we have the following identity:

$$\int_{C^+} Pdx + Qdy = \iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy,$$

where C^+ is the boundary of D oriented in the positive direction. We must compute both integrals and verify that they are the same:

$$\iint\limits_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy = \iint\limits_D (3x^2 + 3y^2) dxdy \underset{\text{polar}}{=} 3\int_0^{2\pi} d\theta \int_0^R r^3 dr = \frac{3\pi R^4}{2}.$$

To evaluate the other integral, we parametrize C^+, the boundary of D: $\sigma(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$,

$$\begin{aligned}
\int_{C^+} Pdx + Qdy &= \int_0^{2\pi} (2R^3\cos^3 t - R^3\sin^3 t, R^3\cos^3 t + R^3\sin^3 t) \cdot (-R\sin t, R\cos t) dt = \\
&= R^4 \int_0^{2\pi} (\sin^4 t + \cos^4 t) dt = R^4 \int_0^{2\pi} \Big(\frac{1-\cos(2t)}{2}\Big)^2 + \Big(\frac{1+\cos(2t)}{2}\Big)^2 = \\
&= R^4 \int_0^{2\pi} \frac{1+\cos^2(2t)}{2} dt = R^4 \int_0^{2\pi} \frac{3+\cos(4t)}{4} dt = \frac{3\pi R^4}{2}.
\end{aligned}$$

PROBLEM 3. (3 p.) Let S be the portion of the surface $x^2 + y^2 = 25$, $y \geq 0$, among the planes $z = 2y$ and $z = 12$. Let W be the solid bounded by the closed surface formed by S and the planes $z = 2y$, $z = 12$ and $y = 0$.

(a) (1 p.) Compute the area of S.

(b) (1 p.) The temperature $T(x, y, z)$ on each point of W is proportional to the distance to the vertical axis. What is the average temperature?

$$\Big(Hint: T_{average} = \frac{\iiint_W T(x,y,z)}{Volume(W)}\Big).$$

(c) (1 p.) Compute the *outer* flux through the surface ∂W (the boundary of W) of the vector field

$$\vec{F}(x, y, z) = (3x + \sin(yz), 2(y - x^2 - z^3), \ln(x^2 + y^2)).$$

Solution:

(a) We parametrize the surface using cylindrical coordinates and fixing the radius to $r = 5$. $\vec{\phi}(\theta, z) = (5\cos\theta, 5\sin\theta, z)$, defined on the set $D = \{(\theta, z) : \theta \in [0, \pi], z \in [10\sin\theta, 12]\}$. Therefore, $\vec{\phi}_\theta \times \vec{\phi}_z = (-5\sin\theta, 5\cos\theta, 0)$ and $\|\vec{\phi}_\theta \times \vec{\phi}_z\| = 5$.
The area of the surface S is:

$$A(S) = \iint_S dS = \iint_D \|\vec{\phi}_\theta \times \vec{\phi}_z\| dz d\theta = \int_0^\pi \int_{10\sin\theta}^{12} 5 dz d\theta = 20(3\pi - 5).$$

(b) We have that $T(x, y, z) = \lambda\, dist((x, y, z), (0, 0, z)) = \lambda\sqrt{x^2 + y^2}$. Therefore,

$$\begin{aligned}
\iiint_W T(x,y,z) dV &= \lambda \iint_{\{x^2+y^2\leq 25,\ y\geq 0\}} \int_{2y}^{12} \sqrt{x^2+y^2} dz\, dy\, dx \\
&\underset{\text{cylindrical}}{=} \lambda \int_0^\pi \int_0^5 \int_{2r\sin\theta}^{12} r^2\, dz\, dr\, d\theta = 5^3\lambda\,(4\pi - 5).
\end{aligned}$$

We have also that

$$Volume(W) = \iiint_W dV = \iint_{\{x^2+y^2\leq 25,\ y\geq 0\}} \int_{2y}^{12} dz\, dy\, dx$$

$$\underset{\text{cylindrical}}{=} \int_0^{\pi} \int_0^{5} \int_{2r\sin\theta}^{12} r\, dz\, dr\, d\theta = 5^2(6\pi - 20/3).$$

Therefore, $T_{average}(x, y, z) = \dfrac{\iiint_W T(x, y, z)}{Volume(W)} = 5\lambda\, \dfrac{4\pi - 5}{6\pi - 20/3}.$

(c) We can apply the Gauss' Theorem since the surface is closed. Hence, the outer flux is

$$\iint\limits_{\partial W} \vec{F} \cdot d\vec{S} = \iiint\limits_{W} div(\vec{F})\, dV = \iiint\limits_{W} 5\, dV = 5\, Volume(W) \underset{\text{part.(b)}}{=} 5^3(6\pi - 20/3).$$