Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

Primer Curso de INGENIERÍA DE TELECOMUNICACIÓN. **CALCULO II**.
Examen 28 de Junio de 2008.

Apellidos... Nombre........................

D.N.I. .... Grupo ....

EXPLAIN CAREFULLY EACH PROBLEM. **Time length: 3.30 hours.**

PROBLEM 1. (3 p.) Given the following limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2}.$$

(a) (1.5 p.) Express it in terms of a definite integral and prove that the following inequalities are satisfied:

$$1 \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2} \le e.$$

(b) (1.5 p.) Show also that is verified the following inequality

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2} \le e-1.$$

Solution:

a) Since $f(x) = e^{x^2}$ is a continuous function on $[0,1]$, there exists the integral

$$I = \int_0^1 e^{x^2}dx.$$

By definition, the infimum of upper sums $\underline{S}$ is equal to the supremum of lower sums $\overline{s}$ and both of them are equal to the value of the integral. Taking the sequence of partitions

$$F_n = \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}, \qquad n \in \mathbb{N}.$$

We have, for each $n \in \mathbb{N}$, the lower and upper sums, respectively,

$$s_n = \frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k-1}{n}\right)^2}, \quad \text{and} \quad S_n = \frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2}.$$

We have taken into account that $f(x) = e^{x^2}$ is an increasing function on the interval $[0,1]$. $f(x) = e^{x^2}$ is continuous on a closed interval, therefore, is also uniformly continuous. So, given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \ge N$, on each interval

$$\Delta_k = \left[\frac{k-1}{n}, \frac{k}{n}\right], \qquad k = 1, \dots, n,$$

we have that

$$e^{\left(\frac{k}{n}\right)^2} - e^{\left(\frac{k-1}{n}\right)^2} \le \frac{\epsilon}{1-0} = \epsilon.$$

Therefore,

$$S_n - s_n = \frac{1}{n}\sum_{k=1}^{n}\left(e^{\left(\frac{k}{n}\right)^2} - e^{\left(\frac{k-1}{n}\right)^2}\right) \le \frac{1}{n}\sum_{k=1}^{n}\epsilon = \epsilon.$$

Which yields

$$\lim_{n\to\infty}(S_n - s_n) = 0,$$

and, as $s_n \le \overline{s} = \underline{S} \le S_n$, then $\lim S_n = \underline{S}$. Finally

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2} = \int_0^1 e^{x^2}dx.$$

Now, for every $x \in [0,1]$, we have $1 \le e^{x^2} \le e$. Therefore,

$$\int_0^1 1dx \le \int_0^1 e^{x^2}dx \le \int_0^1 edx \qquad \Rightarrow \qquad 1 \le \int_0^1 e^{x^2}dx \le e.$$

b) We have also for every $x \in [0,1]$, that $e^{x^2} \le e^x$, so

$$\int_0^1 e^{x^2}dx \le \int_0^1 e^x dx \qquad \Rightarrow \qquad \int_0^1 e^{x^2}dx \le e^1 - e^0 = e - 1.$$

PROBLEM 2. (3 p.) Given the force field in $\mathbb{R}^2$, $\vec{F}(x,y) = (2xe^{x^2} + y + 1, x)$:

(a) (1 p.) Determine if it is a conservative field. In that case, compute a potential function such that $\vec{F}$ is its gradient.

(b) (1 p.) Find the work done by $\vec{F}$ in moving a particle along the curve $\{x^2 + (y+1)^2 = 4,\ y \ge 0\}$, from $(\sqrt{3}, 0)$ to $(-\sqrt{3}, 0)$.

(c) (1 p.) Compute, by means of Green's Theorem, the area of the flat region, D, bounded by the previous curve and the OX axis. That is, compute the following integral

$$\frac{1}{2}\int_{\partial D} xdy - ydx.$$

Solution:

(a)

$$\frac{\partial}{\partial y}(2xe^{x^2} + y + 1) = 1 = \frac{\partial}{\partial x}(x) \Rightarrow$$

it is conservative.

$$\begin{aligned}
\vec{F}(x,y) &= (2xe^{x^2} + y + 1, x) = \vec{\nabla} f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \Rightarrow \\
f &= \int xdy + g(x) = xy + g(x),\ \frac{\partial f}{\partial x} = 2xe^{x^2} + y + 1 = y + g'(x) \Rightarrow \\
g'(x) &= 2xe^{x^2} + 1 \Rightarrow g(x) = \int (2xe^{x^2} + 1)dx = e^{x^2} + x + c \ \Rightarrow \\
f(x,y) &= e^{x^2} + x(y+1) + c.
\end{aligned}$$

(b) Denoting by γ the given oriented curve, since the vector field is conservative, we have that

$$\int_\gamma \vec{F}\cdot d\vec{r} = f(-\sqrt{3},0) - f(\sqrt{3},0) = e^3 - \sqrt{3} + c - (e^3 + \sqrt{3} + c) = -2\sqrt{3}.$$

We can also compute the line integral without using potential function obtained in (a). Since the vector field is conservative, the work does not depend on the trajectory but on the endpoints. Therefore we can use another trajectory with same endpoints, for instance the line segment joining both points. In this case, the parametrization is $\sigma(t) = (t,0)$ $t \in [\sqrt{3}, -\sqrt{3}]$, and $\int_\gamma \vec{F}\cdot d\vec{r} = \int_{\sqrt{3}}^{-\sqrt{3}} (2te^{t^2}+1, t)\cdot(1,0) = -2\sqrt{3}$.

We can compute the work without using the fact that the force field is conservative. In that case, we need to parametrize the curve: $\sigma(t) = (2\cos t, 2\sin t - 1)$, $t \in [\pi/6, 5\pi/6]$,

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_{\pi/6}^{5\pi/6} \left(4\cos t e^{4cos^2 t} + 2\sin t, 2\cos t\right)\cdot(-2\sin t, 2\cos t)dt =$$

$$= \int_{\pi/6}^{5\pi/6} \Big(- 8\cos t \sin t e^{4cos^2 t} + 4\cos(2t)\Big)dt = \Big[e^{4cos^2 t} + 2\sin(2t)\Big]_{\pi/6}^{5\pi/6} = -2\sqrt{3}.$$

(c) We have an oriented closed curve: $\partial D = \gamma_1 + \gamma_2$, where γ_1 is the oriented curve γ given in part (a), and γ_2 is the line segment joining $(\sqrt{3},0)$ with $(-\sqrt{3},0)$, we can parametrize it as $\sigma_2(t) = (t,0)$, $t \in [-\sqrt{3}, \sqrt{3}]$. Thus, the area of D is:

$$\begin{aligned}
A &= \frac{1}{2}\int_{\partial D} xdy - ydx = \frac{1}{2}\int_{\gamma_1} xdy - ydx + \frac{1}{2}\int_{\gamma_2} xdy - ydx = \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-2\sin t + 1, 2\cos t)\cdot(-2\sin t, 2\cos t)dt + \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (0,t)\cdot(1,0)dt = \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4\sin^2 t - 2\sin t + 4\cos^2 t)dt + 0 = \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 2\sin t)dt = \frac{1}{2}\Big[4t + 2\cos t\Big]_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} - \sqrt{3}.
\end{aligned}$$

PROBLEM 3. (3 p.) Let C be the cylindrical surface given by $C := \{(x,y,z) : x^2 + y^2 = 4,\ z \geq 1\}$. Consider the paraboloid's portion $z = x^2 + y^2$ inside the cylinder, let us denote it by S.

(a) (0.75 p.) Compute the area of the paraboloid's surface given above, S.

(b) (0.75 p.) Compute the *inner* flux of the vector field $\vec{F}(x,y,z) = (-xz, -yz, z^2)$ across the paraboloid's surface S.

(c) (1.5 p.) Let $h : \mathbb{R} \to \mathbb{R}$ be a C^1 map, compute the *outer* flux across the paraboloid's surface S, of the following vector field

$$\vec{F}(x,y,z) = \Big(x + e^{-z^2+7y}, \int_0^z \sin(h(t))dt, 0\Big).$$

Solution:

(a) First, we parametrize the paraboloid's surface:
$\vec{\phi}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$, defined in $D = \{r \in [1,2],\ \theta \in [0,2\pi]\}$.

Therefore, $\vec{\phi}_r \times \vec{\phi}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$ and $\|\vec{\phi}_r \times \vec{\phi}_\theta\| = r\sqrt{4r^2+1}$.
The area of the surface S is:

$$A(S) = \iint_S dS = \iint_D \|\vec{\phi}_r \times \vec{\phi}_\theta\| dr d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{4r^2+1}dr$$
$$= 2\pi\frac{2}{3\cdot 8}\Big[(4r^2+1)^{3/2}\Big]_1^2 = \frac{\pi}{6}(17^{3/2} - 5^{3/2}).$$

(b) Using the parametrization done in (a), we obtain the appropriate normal vector: $\vec{\phi}_r \times \vec{\phi}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$.
The inner flux of the vector field $\vec{F}(x,y,z) = (-xz, -yz, z^2)$ is

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_1^2 \vec{F}(\vec{\phi}(r,\theta))\cdot \vec{\phi}_\theta \times \vec{\phi}_r \, dr\, d\theta =$$
$$= \int_0^{2\pi}\int_1^2 (-r^3\cos\theta, -r^3\sin\theta, r^4)\cdot(-2r^2\cos\theta, -2r^2\sin\theta, r)dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 3r^5 dr\, d\theta = 63\pi.$$

(c) Due to the complexity of the vector field, it would be easier to apply Gauss' Theorem. However, the surface is not closed. So, to apply that theorem, we must *close* the surface S adding the bases $S_1 := \{(x,y,z) : x^2+y^2 \le 1 \text{ and } z = 1\}$ and $S_2 := \{(x,y,z) : x^2+y^2 \le 4 \text{ and } z = 4\}$. Now $S \cup S_1 \cup S_2$ bounds a solid region, W, and we can apply Gauss' Theorem, using the fact that

$$div\vec{F} = \frac{\partial}{\partial x}(x + e^{-z^2+7y}) + \frac{\partial}{\partial y}\Big(\int_0^z \sin(h(t))dt\Big) + \frac{\partial}{\partial z}(0) = 1.$$

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S_1} + \iint_{S_2} \vec{F}\cdot d\vec{S_2} \underset{\text{teor. Gauss}}{=} \iiint_W div\vec{F}dV = \iiint_W dV,$$

thus

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W dV - \iint_{S_1} \vec{F}\cdot d\vec{S_1} - \iint_{S_2} \vec{F}\cdot d\vec{S_2},$$

where $\iiint_W dV$ is solved changing to cylindrical coordinates: $(x,y,z) = (r\cos\theta, r\sin\theta, z)$, $JT = r$.
First method: Taking the limits $r \in [0, \sqrt{z}]$, $z \in [1,4]$, $\theta \in [0, 2\pi]$

$$\iiint_W dV = \int_0^{2\pi}\int_1^4\int_0^{\sqrt{z}} r\, dr\, dz\, d\theta = \frac{15\pi}{2}.$$

Second method: Taking the limits $r \in [0,2]$ such that $r \in [0,1]$, $z \in [1,4]$ and $r \in [1,2]$, $z \in [r^2, 4]$; $\theta \in [0, 2\pi]$

$$\iiint_W dV = \int_0^{2\pi}\Big(\int_0^1\int_1^4 r\, dr\, dz + \int_1^2\int_{r^2}^4 r\, dr\, dz\Big)\, d\theta = \frac{15\pi}{2}.$$

Third method: Subtracting paraboloids.

$$\iiint_W dV = \iint_{\{x^2+y^2\le 4\}}\int_{x^2+y^2}^4 dz\, dA - \iint_{\{x^2+y^2\le 1\}}\int_{x^2+y^2}^1 dz\, dA$$
$$\underset{\text{cylindrical}}{=} \int_0^{2\pi}\int_0^2\int_{r^2}^4 rdzdrd\theta - \int_0^{2\pi}\int_0^1\int_{r^2}^1 rdzdrd\theta = \frac{15\pi}{2}.$$

To find $\iint\limits_{S_1} \vec{F} \cdot d\vec{S_1}$ we parametrize S_1 with a similar map of that done in (a), $\vec{\phi}(r,\theta) = (r\cos\theta, r\sin\theta, 1)$, so $\vec{\phi}_r \times \vec{\phi}_\theta = (0,0,r)$. Thus

$$\iint\limits_{S_1} \vec{F} \cdot d\vec{S_1} = \iint\limits_{D_1} \vec{F}(\vec{\phi}(r,\theta)) \cdot \vec{\phi}_\theta \times \vec{\phi}_r dr d\theta = 0.$$

Similarly, we can obtain that

$$\iint\limits_{S_2} \vec{F} \cdot d\vec{S_2} = \iint\limits_{D_2} \vec{F}(\vec{\phi}(r,\theta)) \cdot \vec{\phi}_\theta \times \vec{\phi}_r dr d\theta = 0.$$

Finally,

$$\iint\limits_{S} \vec{F} \cdot d\vec{S} = \iiint\limits_{W} dV = \frac{15\pi}{2}.$$