UNIVERSIDAD CARLOS III DE MADRID

Escuela Politécnica Superior

Departamento de Matemáticas

PROBLEMS, CALCULUS I, 1st COURSE

DEGREES IN:
Technical Industrial Engineering:
Electricity, Industrial Electronics and Mechanics
Technical Telecommunication Engineering:
Sound & Image, Telecommunication Systems and Telematics

CURSO 2008–2009

Set prepared by
Arturo de PABLO
Elena ROMERA

English version by
Marina DELGADO
Javier RODRÍGUEZ

Index

1 Functions of a real variable

1.1 The real line

Problem 1.1.1

i) Consider the real numbers $0 < a < b$, $k > 0$. Prove the inequalities

$$1) \quad a < \sqrt{ab} < \frac{a+b}{2} < b, \qquad 2) \quad \frac{a}{b} < \frac{a+k}{b+k}.$$

ii) Prove that $|a+b| = |a| + |b| \iff ab \geq 0$.

iii) Prove the inequality $|a-b| \geq \Big||a| - |b|\Big|$, for all $a, b \in I\!R$.

iv) Prove that:

$$a)\ \max\{x,y\} = \frac{x+y+|x-y|}{2}, \qquad b)\ \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

v) Express in a single formula the following function $f(x) = (x)_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$.

Problem 1.1.2 Decompose the expressions in n as a product of factors to prove that, for all $n \in I\!\!N$ we have

i) $n^2 - n$ is even;

ii) $n^3 - n$ is a multiple of 6;

iii) $n^2 - 1$ is a multiple of 6 if n is even.

Problem 1.1.3 Use the induction technique to prove the following formulas:

$$i)\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}; \qquad ii)\ \sum_{j=1}^{n}(2j-1) = n^2; \qquad iii)\ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 1.1.4 Prove by induction

i) Geometrical sum: $\displaystyle\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$, for all $n \in I\!\!N,\ r \neq 1$;

ii) Bernouilli inequality: $(1+h)^n \geq 1 + nh$, for all $n \in I\!\!N,\ h > -1$.

Problem 1.1.5 Prove by induction that for all $n \in I\!\!N$ we have

i) $10^n - 1$ is a multiple of 9;

ii) $10^n - (-1)^n$ is a multiple of 11.

Problem 1.1.6

i) Prove that a number is a multiple of 4 if and only if its two last figures make up a multiple of 4.

ii) Prove that a number is a multiple of 2^k if and only if its k last figures make up a multiple of 2^k.

iii) Prove that a number is a multiple of 3 (or 9) if and only if the sum of its figures is a multiple of 3 (or 9). In other terms, $n = \displaystyle\sum_{j=0}^{N} a_j 10^j$ is a multiple of 3 (or 9) if and only if $\displaystyle\sum_{j=0}^{N} a_j$ is a multiple of 3 (or 9).

iv) Prove that a number is a multiple of 11 if and only if the sum of its even placed figures minus the sum of its odd placed figures is a multiple of 11, i.e.: $\displaystyle\sum_{j=0}^{N}(-1)^j a_j$ is a multiple of 11.

Hints: *ii*) write the number in the form $n = 10^k a + b$, with $a \geq 0$, $0 \leq b < 10^k$; *iii*) and *iv*) use problem 1.1.5.

Problem 1.1.7 Prove by induction and using Newton's binomial that for all $n \in I\!\!N$ we have that

i) $n^3 - n$ is a multiple of 6;

ii) $n^5 - n$ is a multiple of 5.

Problem 1.1.8 Prove that:

i) if $n \in \mathbb{N}$ is not a perfect square, $\sqrt{n} \notin \mathbb{Q}$;
ii) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

Hint: *i*) write $n = z^2 r$, where r does not contain any square factor.

Problem 1.1.9 Find the set of $x \in \mathbb{R}$ that verify:

i) $A = \{\,|x-3| \le 8\,\}$, *ii*) $B = \{\,0 < |x-2| < 1/2\,\}$,

iii) $C = \{\,x^2 - 5x + 6 \ge 0\,\}$, *iv*) $D = \{\,x^3(x+3)(x-5) < 0\,\}$,

v) $E = \{\,\dfrac{2x+8}{x^2+8x+7} > 0\,\}$, *vi*) $F = \{\,\dfrac{4}{x} < x\,\}$,

vii) $G = \{\,4x < 2x + 1 \le 3x + 2\,\}$, *viii*) $H = \{\,|x^2 - 2x| < 1\,\}$,

ix) $I = \{\,|x-1||x+2| = 10\,\}$, *x*) $J = \{\,|x-1| + |x-2| > 1\,\}$.

Problem 1.1.10 Given two real numbers $a < b$, let us define, for each $t \in \mathbb{R}$ the number $x(t) = (1-t)a + tb$. Describe the following sets of numbers:

i) $A = \{\,x(t) \,:\, t = 0,\, 1,\, 1/2\,\}$, *ii*) $B = \{\,x(t) \,:\, t \in (0,1)\,\}$,

iii) $C = \{\,x(t) \,:\, t < 0\,\}$, *iv*) $D = \{\,x(t) \,:\, t > 1\,\}$.

Problem 1.1.11 Find the supremum, the infimum, the maximum and the minimum (if they exist) of the following sets of real numbers:

i) $A = \{-1\} \cup [2,3)$;
ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0,1]$;
iii) $C = \{\,x = 2 + 1/n \,:\, n \in \mathbb{N}\,\}$;
iv) $D = \{\,x = (n^2+1)/n \,:\, n \in \mathbb{N}\,\}$;
v) $E = \{\,x \in \mathbb{R} \,:\, 3x^2 - 10x + 3 \le 0\,\}$;
vi) $F = \{\,x \in \mathbb{R} \,:\, (x-a)(x-b)(x-c)(x-d) < 0\,\}$, with $a < b < c < d$ fixed;
vii) $G = \{\,x = 2^{-p} + 5^{-q} \,:\, p, q \in \mathbb{N}\,\}$;
viii) $H = \{\,x = (-1)^n + 1/m \,:\, n, m \in \mathbb{N}\,\}$.

Problem 1.1.12 Represent in $\mathbb{R}^2$ the following sets:

i) $A = \{\,|x-y| < 1\,\}$, *ii*) $B = \{\,x^2 < y < x\,\}$,

iii) $C = \{\,x + y \in \mathbb{Z}\,\}$, *iv*) $D = \{\,|2x| + |y| = 1\,\}$,

v) $E = \{\,(x-1)^2 + (y+2)^2 < 4\,\}$, *vi*) $F = \{\,|1-x| = |y-1|\,\}$,

vii) $G = \{\,4x^2 + y^2 \le 4,\, xy \ge 0\,\}$, *viii*) $H = \{\,1 \le x^2 + y^2 < 9,\, y \ge 0\,\}$.

Problem 1.1.13 Prove that the straight lines $y = mx + b$, $y = nx + c$ are orthogonal if $mn = -1$.

Problem 1.1.14 Consider the plane triangle defined by the points $(a, 0)$, $(-b, 0)$ and $(0, c)$, with $a, b, c > 0$.

i) Compute the intersection point of the three altitudes.

ii) Calculate the intersection point of the three medians.

iii) When do these two points coincide?

Problem 1.1.15

i) Consider the parabola $G = \{y = x^2\}$ and the point $P = (0, 1/4)$. Find $\lambda \in \mathbb{R}$ such that the points of G are equidistant from P and the horizontal line $L = \{y = \lambda\}$.

ii) Conversely, find that the set G such that its points are equidistant from a point $P = (a, b)$ and a straight line $L = \{y = \lambda\}$, is the parabola $y = \alpha x^2 + \beta x + \gamma$. Find α, β, γ.

Problem 1.1.16

i) Find the set of points in the plane such that the sum of their distances to the points $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ is $2a$, $(a > c)$.

ii) Same question, substituting sum by difference (with $a < c$).

1.2 Elementary functions

Problem 1.2.1 Find the domain of the following functions:

$$i)\quad f(x) = \frac{1}{x^2 - 5x + 6}, \qquad ii)\quad f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1},$$

$$iii)\quad f(x) = \frac{1}{x - \sqrt{1 - x^2}}, \qquad iv)\quad f(x) = \sqrt{1 - \sqrt{4 - x^2}},$$

$$v)\quad f(x) = \frac{1}{1 - \log x}, \qquad vi)\quad f(x) = \log(x - x^2),$$

$$vii)\quad f(x) = \frac{\sqrt{5 - x}}{\log x}, \qquad viii)\quad f(x) = \arcsin(\log x).$$

Problem 1.2.2

i) If both f and g are odd functions, what are $f + g$, $f \cdot g$ and $f \circ g$?

ii) What if f is even and g is odd?

Problem 1.2.3 Study the symmetry of the following functions:

$$i)\quad f(x) = \frac{x}{x^2 - 1}, \qquad ii)\quad f(x) = \frac{x^2 - x}{x^2 + 1},$$

$$iii)\quad f(x) = \frac{\sin x}{x}, \qquad iv)\quad f(x) = (\cos x^3)(\sin x^2)\mathrm{e}^{-x^4},$$

$$v)\quad f(x) = \frac{1}{\sqrt{x^2 + 1} - x}, \qquad vi)\quad f(x) = \log(\sqrt{x^2 + 1} - x).$$

Hint: vi) is odd.

Problem 1.2.4 For which numbers $a, b, c, d \in I\!R$ does the function $f(x) = \dfrac{ax+b}{cx+d}$ satisfy $f \circ f = I$ (identity) on the domain of f?

Problem 1.2.5 Check that the function $f(x) = \dfrac{x+3}{1+2x}$ is bijective, defined from $I\!R - \{-1/2\}$ to $I\!R - \{1/2\}$ and find its inverse.

Problem 1.2.6

i) Study which of the following functions are injective, finding their inverses when they have them, or give an example of two points with the same image otherwise.

$$
\begin{array}{llll}
a) & f(x) = 7x - 4, & b) & f(x) = \sin(7x-4), \\
c) & f(x) = (x+1)^3 + 2, & d) & f(x) = \dfrac{x+2}{x+1}, \\
e) & f(x) = x^2 - 3x + 2, & f) & f(x) = \dfrac{x}{x^2+1}, \\
g) & f(x) = \mathrm{e}^{-x}, & h) & f(x) = \log(x+1).
\end{array}
$$

ii) Prove that the function $f(x) = x^2 - 3x + 2$ is injective on $(3/2, \infty)$.

iii) Prove that the function $f(x) = \dfrac{x}{x^2+1}$ is injective on $(1, \infty)$ and find $f^{-1}(\sqrt{2}/3)$.

iv) Study whether the previous functions are surjective and bijective on their domain $D(f)$ in $I\!R$.

Problem 1.2.7 Prove that $a \sin x + b \cos x$ can be written as $A \sin(x + B)$, and find A and B.

Problem 1.2.8 Calculate

$$
\begin{array}{ll}
i) & \operatorname{arctg} \dfrac{1}{2} + \operatorname{arctg} \dfrac{1}{3}, \\
ii) & \operatorname{arctg} 2 + \operatorname{arctg} 3, \\
iii) & \operatorname{arctg} \dfrac{1}{2} + \operatorname{arctg} \dfrac{1}{5} + \operatorname{arctg} \dfrac{1}{8}.
\end{array}
$$

Hint: use the formula for the tangent of a sum and study the signs.

Problem 1.2.9 Simplify the following expressions

$$
\begin{array}{llll}
i) & f(x) = \sin(\arccos x), & ii) & f(x) = \sin(2 \arcsin x), \\
iii) & f(x) = \operatorname{tg}(\arccos x), & iv) & f(x) = \sin(2 \operatorname{arctg} x), \\
v) & f(x) = \cos(2 \operatorname{arctg} x), & vi) & f(x) = \mathrm{e}^{4 \log x}.
\end{array}
$$

Problem 1.2.10 Solve the following system of equations, for $x, y > 0$,

$$
\begin{cases} x^y = y^x \\ y = 3x. \end{cases}
$$

Problem 1.2.11 Describe function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant). Plot them for $f(x) = x^2$ and $f(x) = \sin x$.

i) $g(x) = f(x) + c$, *ii)* $g(x) = f(x + c)$,

iii) $g(x) = f(cx)$, *iv)* $g(x) = f(1/x)$,

v) $g(x) = f(|x|)$, *vi)* $g(x) = |f(x)|$,

vii) $g(x) = 1/f(x)$, *viii)* $g(x) = (f(x))_+ = \max\{f(x), 0\}$.

Problem 1.2.12 Sketch, with as few calculations as possible, the graph of the following functions:

i) $f(x) = (x + 2)^2 - 1$, *ii)* $f(x) = \sqrt{4 - x}$,

iii) $f(x) = x^2 + 1/x$, *iv)* $f(x) = 1/(1 + x^2)$,

v) $f(x) = \min\{x, x^2\}$, *vi)* $f(x) = |\mathrm{e}^x - 1|$,

vii) $f(x) = \sqrt{x - [x]}$, *viii)* $f(x) = 1/[1/x]$,

ix) $f(x) = |x^2 - 1|$, *x)* $f(x) = 1 - \mathrm{e}^{-x}$,

xi) $f(x) = \log(x^2 - 1)$, *xii)* $f(x) = x \sin(1/x)$.

Hint: $[x] = n$ denotes the integer part of x, i.e., the biggest integer $n \leq x$.

Problem 1.2.13 Let us define the hyperbolic functions

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

i) Study their symmetry.

ii) Prove the formulas

a) $\cosh^2 x - \sinh^2 x = 1$, *b)* $\sinh 2x = 2 \sinh x \cosh x$.

iii) Simplify the function $f(x) = \sinh^{-1} x$.

iv) Sketch the graph of the functions $\sinh x$ and $\cosh x$.

Problem 1.2.14 Sketch the following curves given in polar coordinates:

i) $r = 1, \quad \theta \in [0, \pi]$, *ii)* $\theta = 3\pi/4, \quad r \geq 2$,

iii) $r = 2 \sin \theta, \quad \theta \in [0, \pi]$, *iv)* $r = \theta, \quad \theta \in [0, 2\pi]$,

v) $r = \mathrm{e}^\theta, \quad \theta \in [-2\pi, 2\pi]$, *vi)* $r = \sec \theta, \quad \theta \in [0, \pi/2]$,

vii) $r = 1 - \sin \theta, \quad \theta \in [0, 2\pi]$, *viii)* $r = (\cos 2\theta)_+, \quad \theta \in [0, 2\pi]$,

ix) $r = |\cos 2\theta|, \quad \theta \in [0, 2\pi]$, *x)* $r = (\sin 3\theta)_+, \quad \theta \in [0, 2\pi/3]$.

Problem 1.2.15 Sketch the following subsets of the plane given in polar coordinates:

i) $A = \{1 < r < 4\}$, *ii)* $B = \{\pi/6 \le \theta \le \pi/3\}$,

iii) $C = \{r \le \theta,\ 0 \le \theta \le 3\pi/2\}$, *iv)* $D = \{r \le \sec\theta,\ 0 \le \theta \le \pi/4\}$.

1.3 Limits of functions

Problem 1.3.1 Using the ε–δ definition of limit, prove that:

i) $\lim_{x\to 2} x^2 = 4$, *ii)* $\lim_{x\to 3}(5x-1) \ne 16$,

iii) $\lim_{x\to 0} \dfrac{x}{1+\sin^2 x} = 0$, *iv)* $\lim_{x\to 9} \sqrt{x} = 3$.

Problem 1.3.2 Find the following limits simplifying the common factors which might appear:

i) $\lim_{x\to a} \dfrac{x^n - a^n}{x-a}$, *ii)* $\lim_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$,

iii) $\lim_{x\to 64} \dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$, *iv)* $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2}$,

v) $\lim_{h\to 0} \dfrac{\frac{1}{(1-h)^3}-1}{h}$, *vi)* $\lim_{x\to 1} \Big(\dfrac{1}{\sqrt{x}-1} - \dfrac{2}{x-1}\Big)$.

Problem 1.3.3 Using the limits $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$, find the following limits:

i) $\lim_{x\to 0} \dfrac{(\sin 2x^3)^2}{x^6}$, *ii)* $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$,

iii) $\lim_{x\to 0} \dfrac{\operatorname{tg} x^2 + 2x}{x + x^2}$, *iv)* $\lim_{x\to 0} \dfrac{\sin(x+a) - \sin a}{x}$,

v) $\lim_{x\to 0} \dfrac{\log(1+x)}{x}$, *vi)* $\lim_{x\to 0} (1+x)^{1/x}$,

vii) $\lim_{x\to 0} \dfrac{\log(1-2x)}{\sin x}$, *viii)* $\lim_{x\to 0} (1+\sin x)^{2/x}$,

ix) $\lim_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$, *x)* $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3}$,

xi) $\lim_{x\to 0} \Big(\dfrac{x}{\sin x}\Big)^{\frac{\sin x}{\sin x - x}}$, *xii)* $\lim_{x\to 0} (\cos x)^{1/x^2}$,

xiii) $\lim_{x\to \pi} \dfrac{1-\sin(x/2)}{(x-\pi)^2}$, *xiv)* $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

Hint: it may be necessary to use a change of variables and the limit of the composite function.

Problem 1.3.4 Calculate the following limits:

$$i)\quad \lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}, \qquad ii)\quad \lim_{x\to\infty} \frac{x+\sin x^3}{5x+6},$$

$$iii)\quad \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}, \qquad iv)\quad \lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right),$$

$$v)\quad \lim_{x\to\infty} \frac{e^x}{e^x-1}, \qquad vi)\quad \lim_{x\to-\infty} \frac{e^x}{e^x-1},$$

$$vii)\quad \lim_{x\to\infty} \frac{x-2}{\sqrt{4x^2+1}}, \qquad viii)\quad \lim_{x\to-\infty} \frac{x-2}{\sqrt{4x^2+1}}.$$

Problem 1.3.5 Find the one-sided limits:

$$i)\quad \lim_{t\to0^+} \left(\frac{1}{t}\right)^{[t]}, \qquad ii)\quad \lim_{t\to0^-} \left(\frac{1}{t}\right)^{[t]},$$

$$iii)\quad \lim_{t\to0^+} e^{1/t}, \qquad iv)\quad \lim_{t\to0^-} e^{1/t}.$$

Problem 1.3.6 Find the limits

$$i)\ \lim_{x\to-\infty} \left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}, \qquad ii)\ \lim_{t\to0} \frac{1-e^{1/t}}{1+e^{1/t}}$$

Problem 1.3.7

i) Establish the relation between a and b so that

$$\lim_{x\to1} x^{a/(1-x)} = \lim_{x\to0} (\cos x)^{b/x^2}.$$

ii) If $f(x) = \log(\log x)$ and $\alpha > 0$, find $\lim_{x\to\infty} (f(x) - f(\alpha x))$ and $\lim_{x\to\infty} (f(x) - f(x^\alpha))$.

Problem 1.3.8

i) Prove that if $\lim_{x\to0} f(x) = 0$ then $\lim_{x\to0} f(x)\sin 1/x = 0$.

ii) Calculate $\lim_{x\to0} \dfrac{x}{2+\sin 1/x}$.

1.4 Continuity

Problem 1.4.1

i) Prove that if f is continuous at a point a and g is at $f(a)$, then $g\circ f$ is continuous at a.

ii) Prove that if f is continuous, then $|f|$ is also. Is the reciprocal true?

iii) What can be said of a function that only takes values on $\mathbb{Q}$?

Problem 1.4.2 Find $\lambda \in \mathbb{R}$ so that the function $b(x) = \dfrac{1}{\lambda x^2 - 2\lambda x + 1}$ is continuous on:

$$i)\ \mathbb{R}, \qquad ii)\ [0,1].$$

Problem 1.4.3 Study the continuity of the following functions:

i) $f(x) = \dfrac{e^{-5x} + \cos x}{x^2 - 8x + 12}$;

ii) $g(x) = e^{3/x} + x^3 - 9$;

iii) $h(x) = x^3 \operatorname{tg}(3x + 2)$;

iv) $j(x) = \sqrt{x^2 - 5x + 6}$;

v) $k(x) = (\arcsin x)^3$;

vi) $m(x) = (x - 5)\log(8x - 3)$;

Problem 1.4.4 Study the continuity of the following functions:

i) $f(x) = x - [x]$;

ii) $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0; \end{cases}$

iii) $f(x) = \begin{cases} \dfrac{\operatorname{tg} x}{\sqrt{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ e^{1/x} & \text{if } x < 0; \end{cases}$

iv) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}. \end{cases}$

Problem 1.4.5 Prove the following fixed points theorems:

i) Let $f : [0,1] \longrightarrow [0,1]$ a continuous function. Then there exists $c \in [0,1]$ such that $f(c) = c$.

ii) Let $f,\ g : [a,b] \longrightarrow \mathbb{R}$ two continuous functions such that $f(a) > g(a),\ f(b) < g(b)$. Then there exists $c \in (a,b)$ such that $f(c) = g(c)$.

2 Differential calculus in one variable

2.1 Derivatives

Problem 2.1.1 Let f, g be differentiable functions on all $\mathbb{R}$. Write down the derivative of the following functions on their domain:

$$i) \quad h(x) = \sqrt{f^2(x) + g^2(x)}, \qquad ii) \quad h(x) = \operatorname{arctg}\left(\frac{f(x)}{g(x)}\right),$$

$$iii) \quad h(x) = f(g(x))e^{f(x)}, \qquad iv) \quad h(x) = \log(g(x)\sin(f(x))),$$

$$v) \quad h(x) = (f(x))^{g(x)}, \qquad vi) \quad h(x) = \frac{1}{\log(f(x) + g^2(x))}.$$

Problem 2.1.2

i) Build a continuous function for all $I\!R$ that vanishes for $|x| \geq 2$ and takes value one for $|x| \leq 1$.

ii) Build another one which is also differentiable.

Problem 2.1.3 Starting from the hyperbolic functions $\sinh x$ and $\cosh x$, we define $\operatorname{tgh} x = \dfrac{\sinh x}{\cosh x}$ and $\operatorname{sech} x = \dfrac{1}{\cosh x}$. Prove the formulas

$$i)\quad (\sinh x)' = \cosh x, \qquad ii)\quad (\cosh x)' = \sinh x,$$

$$iii)\quad (\operatorname{tgh} x)' = \operatorname{sech}^2 x, \qquad iv)\quad (\operatorname{sech} x)' = -\operatorname{sech} x \operatorname{tgh} x.$$

Problem 2.1.4 Check that the following functions fulfill the specified differential equations, where c, c_1 and c_2 are constants.

$$i)\quad f(x) = \frac{c}{x}, \qquad xf' + f = 0;$$

$$ii)\quad f(x) = x \operatorname{tg} x, \qquad xf' - f - f^2 = x^2;$$

$$iii)\quad f(x) = c_1 \sin 3x + c_2 \cos 3x, \qquad f'' + 9f = 0;$$

$$iv)\quad f(x) = c_1 e^{3x} + c_2 e^{-3x}, \qquad f'' - 9f = 0;$$

$$v)\quad f(x) = c_1 e^{2x} + c_2 e^{5x}, \qquad f'' - 7f' + 10f = 0;$$

$$vi)\quad f(x) = \log(c_1 e^{x} + e^{-x}) + c_2, \qquad f'' + (f')^2 = 1.$$

Problem 2.1.5 Prove the identities

$$i)\quad \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \frac{\pi}{2}, \qquad x > 0;$$

$$ii)\quad \operatorname{arctg} \frac{1+x}{1-x} - \operatorname{arctg} x = \frac{\pi}{4}, \qquad x < 1;$$

$$iii)\quad 2 \operatorname{arctg} x + \arcsin \frac{2x}{1+x^2} = \pi, \qquad x \geq 1.$$

Hint: differentiate and substitute at some point of the interval. The result is *not* true outside the specified intervals.

Problem 2.1.6 Find the value of $a \in I\!R$ for which the parabola $f(x) = ax^2$ is tangent to the graph of $g(x) = \log x$, and write the equation of the common tangent.

Problem 2.1.7 Find the points at which the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent.

Problem 2.1.8 Find the angle spanned by the left and right tangents at the origin to the plot of the function

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Problem 2.1.9 Given the function

$$f(x) = \begin{cases} (3-x^2)/2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1, \end{cases}$$

i) study its continuity and differentiability;

ii) may we apply the Mean Value Theorem on [0,2]? If the answer is positive, find the point (or points) from the thesis of the theorem.

Problem 2.1.10 Study the continuity and differentiability of the function

$$f(x) = \sqrt{x+2}\,\arccos(x+2).$$

Problem 2.1.11 Find the minimum value of a for which the function $f(x) = |\alpha x^2 - x + 3|$ is differentiable on all $\mathbb{R}$.

Problem 2.1.12 The function $f(x) = 1 - x^{2/3}$ vanishes at -1 and 1 and, nonetheless, $f'(x) \neq 0$ on $(-1,1)$. Explain this apparent contradiction with the Rolle's Theorem.

Problem 2.1.13 Given the function $f(x) = \begin{cases} a + bx^2 & \text{if } |x| \leq c \\ |x|^{-1} & \text{if } |x| > c \end{cases}$, with $c > 0$, find a and b so it is continuous and differentiable on the whole of $\mathbb{R}$.

Problem 2.1.14 Using the Mean Value Theorem, approximate $26^{2/3}$ and $\log(3/2)$.

Problem 2.1.15

i) If f is a differentiable function, find

$$\lim_{h\to 0} \frac{f(a+ph) - f(a-qh)}{h}.$$

ii) Can the previous limit exist for a function which is not differentiable at $x = a$?

iii) If f is an even and differentiable function, find $f'(0)$.

iv) If f is a twice differentiable function, find

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Problem 2.1.16 Find the limits of the problems 1.3.2 and 1.3.3 using L'Hôpital's Rule, writing them previously in the appropriate form.

Problem 2.1.17 Find the following limits:

i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$, *ii)* $\displaystyle\lim_{x\to 0} \frac{\log|\sin 7x|}{\log|\sin x|}$,

iii) $\displaystyle\lim_{x\to 1} \log x \cdot \log(x-1)$, *iv)* $\displaystyle\lim_{x\to\infty} x^{1/x}$,

v) $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$, *vi)* $\displaystyle\lim_{x\to\infty} x\Big(\operatorname{tg}(2/x) - \operatorname{tg}(1/x)\Big)$.

Problem 2.1.18 Find the following limits:

i) $\lim_{x\to\infty} \frac{x^{x-1}}{(x-1)^x}$, *ii)* $\lim_{x\to 0} \frac{1+\sin x - \mathrm{e}^x}{\operatorname{arctg} x}$,

iii) $\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3}$, *iv)* $\lim_{x\to 0}(1+x^2)^{3/(2\arcsin x)}$,

v) $\lim_{x\to 1/2}(2x^2+3x-2)\operatorname{tg}(\pi x)$, *vi)* $\lim_{x\to 0} \frac{2x\sin x}{\sec x - 1}$,

vii) $\lim_{x\to -\infty}(\sqrt{x^2+1}-\sqrt{x^2-4x})$, *viii)* $\lim_{x\to 0^+} x^{1/\log x}$.

Problem 2.1.19 Let h be a twice differentiable function, and let

$$f(x) = \begin{cases} h(x)/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Assuming that f is continuous, find $h(0)$, $h'(0)$ and $h''(0)$.

Problem 2.1.20 Find a so that $\lim_{x\to 0} \frac{\mathrm{e}^{ax}-\mathrm{e}^x - x}{x^2}$ is finite and find the value of the limit.

Problem 2.1.21 Calculate the following limits:

i) $\lim_{x\to\infty} x\Big((1+1/x)^x - \mathrm{e}\Big)$, *ii)* $\lim_{x\to\infty} \frac{(1+1/x)^{x^2}}{\mathrm{e}^x}$,

iii) $\lim_{x\to\infty}\left(\frac{2^{1/x}+18^{1/x}}{2}\right)^x$, *iv)* $\lim_{x\to\infty}\left(\frac{1}{p}\sum_{i=1}^{p} a_i^{1/x}\right)^x$, $p \in I\!N$, $a_i > 0$.

Problem 2.1.22 Given a differentiable function f, which satisfies $\lim_{x\to 0} \frac{f(2x^3)}{5x^3} = 1$,

i) justify that $f(0) = 0$;

ii) prove that $f'(0) = 5/2$;

iii) calculate $\lim_{x\to 0} \frac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 2.1.23 Using the Mean Value Theorem compute the limit

$$\lim_{x\to\infty}\left[(1+x)^{1+\frac{1}{1+x}} - x^{1+\frac{1}{x}}\right].$$

Problem 2.1.24

i) Let $f(x) = \sin x$. Calculate the values of x for which we have $(f^{-1})'(x) = 5/4$.

ii) Same question with $g(x) = \log(x+\sqrt{x^2+1})$ and $(g^{-1})'(x) = 2$.

Problem 2.1.25 The equation

$$\begin{cases} \mathrm{e}^{-f} f' = 2 + \operatorname{tg} x \\ f(0) = 1, \end{cases}$$

defines a differentiable one-to-one function f on the interval $[-\pi/4, \pi/4]$. We define the function $g(x) = f^{-1}(x+1)$. Find the limit

$$\lim_{x \to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-\sin x}}{g(x)}.$$

Problem 2.1.26

i) Let $f : [a,b] \longrightarrow I\!R$, differentiable. If f has $k \geq 2$ roots on $[a,b]$, then f' has at least $k-1$ roots on $[a,b]$.

ii) If f is n times differentiable on $[a,b]$ and vanishes at $n+1$ different points on $[a,b]$, prove that $f^{(n)}$ vanishes at least once on $[a,b]$.

Problem 2.1.27 Calculate how many different solutions do the following equations have on the given intervals:

$i)$	$x^7 + 4x = 3,$	on $I\!R$;	$ii)$	$x^5 = 5x - 6,$	on $I\!R$;
$iii)$	$x^4 - 4x^3 = 1,$	on $I\!R$;	$iv)$	$\sin x = 2x - 1,$	on $I\!R$;
$v)$	$x^x = 2,$	on $[1, \infty)$;	$vi)$	$x^2 = \log(1/x),$	on $(1, \infty)$.

2.2 Extrema of functions

Problem 2.2.1 Let the function $f(x) = |x^3(x-4)| - 1$.

i) Study its continuity and differentiability.

ii) Find its local extrema.

iii) Prove that the equation $f(x) = 0$ has a single solution on the interval $[0,1]$.

Problem 2.2.2 A tomato sauce company wants to manufacture cylindrical cans of fixed volume V. What must be the relation between the radius r of the basis and its height so the minimum amount of material is employed?

Problem 2.2.3 Find the area of the rectangle, with sides parallel to the axes and inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$ with maximum area.

Problem 2.2.4 Find the area of the triangle formed by a tangent to the parabola $y = 6 - x^2$ and the positive semiaxes which has maximum area.

Problem 2.2.5 A right triangle ABC has the A vertex at the origin, B on the circumference $(x-1)^2 + y^2 = 1$, and the side AC on the horizontal axis. Calculate C such that the area of the triangle is maximum.

Problem 2.2.6 Let $P = (x_0, y_0)$ be a point in the first quadrant. Draw a straight line which passes through P and cuts the axes at $A = (x_0+\alpha, 0)$ and $B = (0, y_0+\beta)$ respectively. Calculate $\alpha, \beta > 0$ such that the following magnitudes are minimized:

i) length of AB;

ii) length of OA plus OB;

iii) area of the triangle OAB.

Hint: $\beta = x_0 y_0/\alpha$.

Problem 2.2.7

i) Prove Bernoulli's inequality: $(1+x)^a \geq 1 + ax$, for all $a \geq 1$, $x > -1$.

ii) Prove that $\mathrm{e}^x \geq 1 + x$ for all $x \in \mathbb{R}$.

iii) Prove that $\dfrac{x}{1+x} \leq \log(1+x) \leq x$ for all $x > -1$.

Hint: minimize the appropriate functions.

Problem 2.2.8 Prove that if $a > 0$, the function $f(x) = \left(1 + \dfrac{a}{x}\right)^x$ increases, for $x > 0$, from 1 up to e^a.
Hint: use the previous estimates.

Problem 2.2.9 Prove by induction the inequality $n^n < n!\,\mathrm{e}^n$, for all $n \in \mathbb{N}$.

Problem 2.2.10

i) Prove that $\dfrac{\log x}{x} < \dfrac{1}{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

ii) Find as a conclusion that $\mathrm{e}^x > x^{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

Problem 2.2.11 Find the absolute maxima and minima of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ on the interval $[-2, 1]$.

2.3 Graphical representation

Problem 2.3.1 Prove that if f and g are convex, twice differentiable functions, and f is increasing, then $h = f \circ g$ is convex.

Problem 2.3.2

i) Sketch the plot of the function $f(x) = x + \log|x^2 - 1|$.

ii) From it, draw the plot of the functions

$$a)\quad g(x) = |x| + \log|x^2-1|, \qquad b)\quad h(x) = \Big|x + \log|x^2-1|\Big|.$$

Problem 2.3.3 Represent graphically the following functions:

i) $y = \mathrm{e}^x \sin x$, *ii)* $y = \sqrt{x^2 - 1} - 1$, *iii)* $y = x\mathrm{e}^{1/x}$,

iv) $y = x^2 \mathrm{e}^x$, *v)* $y = (x-2)x^{2/3}$, *vi)* $y = (x^2 - 1)\log \dfrac{1+x}{1-x}$,

vii) $y = \dfrac{x}{\log x}$, *viii)* $y = \dfrac{x^2-1}{x^2+1}$, *ix)* $y = \dfrac{\mathrm{e}^{1/x}}{1-x}$,

x) $y = \log[(x-1)(x-2)]$, *xi)* $y = \dfrac{\mathrm{e}^x}{x(x-1)}$, *xii)* $y = 2\sin x + \cos 2x$,

xiii) $y = \dfrac{x-2}{\sqrt{4x^2+1}}$, *xiv)* $y = \sqrt{|x-4|}$, *xv)* $y = \dfrac{1}{1+\mathrm{e}^x}$,

xvi) $y = \dfrac{\mathrm{e}^{2x}}{\mathrm{e}^x - 1}$, *xvii)* $y = \mathrm{e}^{-x}\sin x$, *xviii)* $y = x^2 \sin(1/x)$.

Problem 2.3.4 Represent graphically the following functions:

i) $f(x) = \min\{\log|x^3 - 3|, \log|x+3|\}$, *ii)* $g(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$,

iii) $h(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $a > 0$, *iv)* $k(x) = x\sqrt{|x^2-1|}$,

v) $p(x) = \operatorname{arctg}(\log(|x^2-1|))$, *vi)* $w(x) = 2\operatorname{arctg} x + \arcsin\Big(\dfrac{2x}{1+x^2}\Big)$.

Problem 2.3.5 Draw the graph of the function

$$f(x) = \frac{e^{1/x}}{1+x}, \qquad x \neq 0; \qquad f(0) = 0.$$

and study in a reasoned way how many solution does the equation $f(x) = x^3$ have on $I\!R$.

Problem 2.3.6 Given the function $f(x) = \dfrac{1+x}{3+x^2}$, represent the graph of the functions

$$i) \quad g(x) = \sup_{y>x} f(y), \qquad ii) \quad h(x) = \inf_{y>x} f(y).$$

Problem 2.3.7

i) Calculate the image of the function $f(x) = 1 + (\operatorname{arctg} x)^2$.

ii) Calculate the values of $A \in I\!R$ such that the function

$$g(x) = \frac{1}{A + \log f(x)},$$

is continuous on all $I\!R$.

iii) Find the supremum and infimum of g if $A = 1$.

iv) Sketch the graph of g in this last case.

Problem 2.3.8 We consider the function $f(x) = \log(1 + x^2)$.

i) Calculate the tangent lines at its inflection points, and sketch the graph of f and those lines.

ii) Prove that the function $g(x) = \max\{\, f(x),\ |x| + \alpha \,\}$ verifies the hypothesis of the Mean Value Theorem on any interval $[a, b] \subset I\!R$ if and only if $\alpha = \log 2 - 1$.

iii) For the previous value of α, obtain the point or points whose existence is guaranteed by the aforementioned theorem applied to function g on the interval $[-1, 2]$.

2.4 Taylor polynomial

Problem 2.4.1 Write the Taylor polynomial of order 5 around the origin for the following functions:

$$i)\quad e^x \sin x, \qquad ii)\quad e^{-x^2}\cos 2x, \qquad iii)\quad \sin x \cos 2x,$$

$$iv)\quad e^x \log(1-x), \qquad v)\quad (\sin x)^2, \qquad vi)\quad \frac{1}{1-x^3}.$$

Problem 2.4.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

Problem 2.4.3 Write the Taylor polynomial of order n around the given point, for the following functions:

i) $f(x) = 1/x$ at $a = -1$;

ii) $f(x) = xe^{-2x}$ at $a = 0$;

iii) $f(x) = (1 + e^x)^2$ at $a = 0$.

Problem 2.4.4 Prove the formulas

$$i)\quad \sin x = o(x^\alpha), \quad \forall\, \alpha < 1, \qquad \text{when } x \to 0;$$

$$ii)\quad \log(1 + x^2) = o(x), \qquad \text{when } x \to 0;$$

$$iii)\quad \log x = o(x), \qquad \text{when } x \to \infty;$$

$$iv)\quad \operatorname{tg} x - \sin x = o(x^2), \qquad \text{when } x \to 0.$$

Problem 2.4.5 Prove the formulas

$$i)\quad \sqrt{1+x} = 1 + \frac{x}{2} + o(x), \qquad \text{when } x \to 0;$$

$$ii)\quad \sin(o(x)) = o(x), \qquad \text{when } x \to 0;$$

$$iii)\quad \sin(f(x) + o(x)) = \sin(f(x)) + o(x), \qquad \text{when } x \to 0.$$

Hint: *ii*) means that if g is such that $\lim_{x\to 0} \dfrac{g(x)}{x} = 0$, then $\lim_{x\to 0} \dfrac{\sin(g(x))}{x} = 0$.

Problem 2.4.6 Find the following limits using Taylor's Theorem:

i) $\lim_{x\to 0} \dfrac{e^x - \sin x - 1}{x^2}$,

ii) $\lim_{x\to 0} \dfrac{\sin x - x + x^3/6}{x^5}$,

iii) $\lim_{x\to 0} \dfrac{\cos x - \sqrt{1-x}}{\sin x}$,

iv) $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3}$,

v) $\lim_{x\to 0} \dfrac{x - \sin x}{x(1-\cos 3x)}$,

vi) $\lim_{x\to 0} \dfrac{\cos x + e^x - x - 2}{x^3}$,

vii) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$,

viii) $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{x} - \cot x\right)$,

ix) $\lim_{x\to\infty} x^{3/2}(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x})$,

x) $\lim_{x\to\infty} \left[x - x^2 \log(1+1/x)\right]$.

Problem 2.4.7 Calculate the Taylor polynomial of order 4 at the origin for the function $f(x) = 1 + x^3 \sin x$ and decide whether f has at that point a maximum, a minimum or an inflection point.

Problem 2.4.8

i) Calculate approximately the value of $\dfrac{1}{\sqrt{1.1}}$ using a Taylor polynomial of degree 3. How much is the error?

ii) Approximate $\sqrt[3]{28}$ using a Taylor polynomial of degree 2. Evaluate the error.

Problem 2.4.9

i) Approximate the function $f(x) = \cos x + e^x$ through a third order polynomial around the origin.

ii) Estimate the error when the previous approximation is used at $x \in [-1/4, 1/4]$.

Problem 2.4.10 How many terms should be taken in Taylor expansion around the origin for the function $f(x) = e^x$ in order to obtain a polynomial which approximates it on $[-1, 1]$ with three exact significant figures?

3 Sequences and series

3.1 Real numbers sequences

Problem 3.1.1

i) Let $\{x_n\}$ be a convergent sequence and $\{y_n\}$ a divergent one; What can we say about the product sequence $\{x_n y_n\}$, sum sequence $\{x_n + y_n\}$, and quotient sequence $\{y_n/x_n\}$ (if $x_n \neq 0$ for all $n \in \mathbb{N}$)?

ii) Prove that if $\{x_n\}$ is convergent, then the sequence $\{|x_n|\}$ is also convergent. Is the reciprocal true?

iii) What can we say about a sequence of integer numbers that is convergent?

iv) Show that every convergent sequence is bounded.

Problem 3.1.2 Given the following sequences in a recurrent way, write down the general term and compute the limit.

$$i)\ a_0 = 0, \quad a_{n+1} = \frac{a_n + 1}{2}; \qquad ii)\ b_0 = 1, \quad b_{n+1} = \sqrt{2b_n}\,.$$

Problem 3.1.3 Compute the following limits:

$$i) \quad \lim_{n\to\infty} \sqrt[n]{a}, \quad (a > 0), \qquad ii) \quad \lim_{n\to\infty} n^{-3/n},$$

$$iii) \quad \lim_{n\to\infty} \sqrt[n]{a^n + b^n}, \quad (a, b > 0), \qquad iv) \quad \lim_{n\to\infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n, \quad (a, b > 0),$$

$$v) \quad \lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right), \qquad vi) \quad \lim_{n\to\infty} (\sqrt[4]{n^2+1} - \sqrt{n+1}),$$

$$vii) \quad \lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}, \qquad viii) \quad \lim_{n\to\infty} \left(\frac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}.$$

Problem 3.1.4 Calculate the following limits:

$$i) \quad \lim_{n\to\infty} \frac{n}{\pi} \sin n\pi, \qquad ii) \quad \lim_{n\to\infty} \frac{n(\mathrm{e}^{1/n} - \mathrm{e}^{\sin 1/n})}{1 - n \sin 1/n},$$

$$iii) \quad \lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\log n}, \qquad iv) \quad \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}},$$

$$v) \quad \lim_{n\to\infty} \frac{2^n}{n!}, \qquad vi) \quad \lim_{n\to\infty} \frac{n^2}{2^n},$$

$$vii) \quad \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^n}, \qquad viii) \quad \lim_{n\to\infty} \frac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \cdots + n\sqrt[n]{n}}{n^2}.$$

Problem 3.1.5 Compute the following limits:

$$i)\ \lim_{n\to\infty} \left(\cos\frac{b}{n} + a \sin\frac{b}{n}\right)^n; \qquad ii)\ \lim_{n\to\infty} \sqrt[u_n]{\frac{a - bu_n}{a + u_n}}, \quad \text{if} \quad \lim_{n\to\infty} u_n = 0,\ a > 0.$$

Problem 3.1.6 Compute the following limits:

$$i)\ \lim_{n\to\infty} \frac{\displaystyle\sum_{k=1}^{n} \sin\frac{\pi}{k}}{\log n}, \qquad ii)\ \lim_{n\to\infty} \prod_{k=1}^{n} (2k-1)^{1/n^2}, \qquad iii)\ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \sin\frac{1}{k}.$$

Problem 3.1.7 If $\lim_{n\to\infty} a_n = \ell$, find

$$\lim_{n\to\infty} \frac{a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}}{\log(n+1)}.$$

Problem 3.1.8 Let $\{a_n\}$ be a sequence of positive terms verifying $\lim_{n\to\infty}(a_n - n) = L$.

i) Show that $\lim_{n\to\infty} \frac{a_n}{n} = 1$.

ii) Prove that $\lim_{n\to\infty} n\log(a_n/n) = L$.

Problem 3.1.9 Let $\{a_n\}$ be a sequence of positive numbers verifying $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$. Compute, by means of Stolz's test, the limit

$$\lim_{n\to\infty} \sqrt[n^2]{\frac{a_n^n}{a_1 \cdot a_2 \cdots a_n}}.$$

Problem 3.1.10 Prove the formulas, for $n \to \infty$,

$$i)\ \sin(\pi n + o(1)) = o(1), \qquad ii)\ \sin(\pi\sqrt{n^2+1}) = (-1)^n \sin\left(\frac{\pi}{2n}\right) + o\left(\frac{1}{n}\right).$$

Hint: use problem 2.4.5.

Problem 3.1.11 Prove that the following sequences are monotonic, analyze if they are bounded, and compute the limits if they exist.

$$\begin{array}{llll} i) & \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots & ii) & \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots \\ iii) & u_{n+1} = 3 + \frac{u_n}{2}, \quad u_0 = 0. & iv) & u_{n+1} = 3 + 2u_n, \quad u_0 = 0. \\ v) & u_{n+1} = \frac{u_n^3 + 6}{7}, & & a)\ u_0 = 1/2, \quad b)\ u_0 = 3/2, \quad c)\ u_0 = 3. \end{array}$$

Problem 3.1.12 Consider the sequence defined as $a_{n+1} = \sqrt{1+3a_n} - 1,\ a_0 = 1/2$.

i) Prove that it converges and compute its limit.

ii) Compute $\lim_{n\to\infty} \frac{a_{n+1}-1}{a_n - 1}$.

Problem 3.1.13 Let the sequence be defined as $b_{n+1} = 1 - \frac{b_n}{2}$, with $b_0 = 0$.

i) Show that is an alternating sequence, that is, $\text{sign}(b_{n+1} - b_n) = -\text{sign}(b_n - b_{n-1})$.

ii) Compute the limit ℓ if it exists.

iii) Show that $|b_{n+1} - \ell| = \frac{1}{2}|b_n - \ell|$.

iv) Prove that $\lim_{n\to\infty} b_n = \ell$.

Hint: *iii)* $|b_n - \ell| = (\frac{1}{2})^n\,|b_0 - \ell|$.

Problem 3.1.14 Consider the sequence defined by $c_{n+1} = f(c_n)$, where $f(x) = \frac{1}{1+x}$, $c_0 = 0$. Prove that converges by means of the following steps:

i) Compute the limit ℓ if it exists.

ii) Prove that if $x \in [1/2, 1]$, then $f(x) \in [1/2, 1]$.

iii) Prove that $c_n \in [1/2, 1]$ for all $n \geq 1$.

iv) Show that $|f'(x)| \leq k < 1$ for every $x \in [1/2, 1]$. This implies that $|c_{n+1} - \ell| \leq k^n |c_1 - \ell| \to 0$.

Problem 3.1.15

i) Use the technique of the previous problem for the sequence

$$d_0 = \frac{1}{2}, \quad d_{n+1} = 2 + \frac{4}{d_n}, \; n \geq 0,$$

on the interval $[3, 10/3]$.

ii) Compute $\lim_{n\to\infty} \dfrac{d_{n+1} - \ell}{d_n - \ell}$.

Problem 3.1.16 Consider the sequence of real numbers defined recurrently

$$x_1 = 1, \qquad x_n = \frac{x_{n-1}(1 + x_{n-1})}{1 + 2x_{n-1}}.$$

Prove that converges and compute its limit.

Problem 3.1.17 Describe the behaviour of the recurrent sequences of the previous problems, sketching in two cartesian axes each pair of consecutive terms (*spider's web diagram*).

3.2 Series of real numbers

Problem 3.2.1 Analyze the convergence of the following series of positive terms:

i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^n$, *ii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^2}$, *iii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^4+1}}$,

iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$, *v)* $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+n}$, *vi)* $\displaystyle\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$,

vii) $\displaystyle\sum_{n=1}^{\infty} \arcsin(\frac{1}{\sqrt{n}})$, *viii)* $\displaystyle\sum_{n=1}^{\infty} \frac{3n-1}{(\sqrt{2})^n}$, *ix)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$,

x) $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$, *xi)* $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} 3^{-n}$, *xii)* $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \mathrm{e}^{-n}$,

xiii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$, *xiv)* $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(\log n)^n}$, *xv)* $\displaystyle\sum_{n=2}^{\infty} [\sqrt{n^2+1} - n]$,

xvi) $\displaystyle\sum_{n=2}^{\infty} \log(\frac{n+1}{n})$, *xvii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$, *xviii)* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$.

Hints: (in general it can be applied more than one test); *i*), *viii*), *x*), *xi*), *xiii*), *xiv*), root's test; *ix*), quotient's test; *ii*), *iii*), *iv*), *v*), *vi*), *vii*), *xv*), *xvi*), *xvii*), *xviii*), comparison; *xii*), compute the limit (see 2.1.21 *ii*)).

Problem 3.2.2 Prove that the series

$$\sum_{n=1}^{\infty}\Big(\frac{a}{2n-1}-\frac{b}{2n+1}\Big)$$

is convergent if and only if $a=b$.

Problem 3.2.3

i) Analyze the convergence of the series $\sum_{n=1}^{\infty} n(1+a)^n \mathrm{e}^{-an}$, for different values of $a>-1$.

ii) Do the same for the series $\sum_{n=1}^{\infty}\frac{n^n}{a^n n!}$, for different values of $a>0$.

iii) Do the same for the series $\sum_{n=1}^{\infty}\frac{n!\mathrm{e}^n}{n^{n+a}}$, for different values of $a\in\mathbb{R}$.

Hints: *ii)* and *iii)* use Stirling's formula.

Problem 3.2.4 Analyze the absolute and conditional convergence of the following alternating series:

i) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$, *ii)* $\sum_{n=2}^{\infty}\sin(\pi n+1/n)$,

iii) $\sum_{n=1}^{\infty}(-1)^n(\operatorname{arctg}1/n)^2$, *iv)* $\sum_{n=1}^{\infty}(-1)^n(\operatorname{arctg}n)^2$,

v) $\sum_{n=1}^{\infty}(-1)^n[\sqrt{n^2-1}-n]$, *vi)* $\sum_{n=1}^{\infty}(-1)^n\log(\frac{n}{n+1})$,

vii) $\sum_{n=1}^{\infty}(-1)^n(1-\cos(1/n))$, *viii)* $\sum_{n=1}^{\infty}\frac{(-1)^n}{\log(\mathrm{e}^n+\mathrm{e}^{-n})}$.

Problem 3.2.5 Use the Taylor expansion of the function $\operatorname{arctg} x$ to study the convergence of the series

$$\sum_{n=1}^{\infty}\Big(\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}\Big).$$

Problem 3.2.6 Compute how many terms are necessary to approximate the following sums with an error less than 10^{-3}:

i) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$, *ii)* $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Problem 3.2.7 Compute the sum of the following series:

i) $\sum_{n=0}^{\infty}\frac{3^{n+1}-2^{n-3}}{4^n}$, *ii)* $\sum_{n=1}^{\infty}\frac{n}{2^n}$, *iii)* $\sum_{n=0}^{\infty}\frac{4n+1}{3^n}$,

iv) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}$, *v)* $\sum_{n=1}^{\infty}\log\Big[\frac{n(n+2)}{(n+1)^2}\Big]$.

Problem 3.2.8 Compute the sum of the following series:

$$i) \quad \sum_{n=0}^{\infty} a^{[n/2]} b^{[(n+1)/2]}, \quad (|ab| < 1), \qquad ii) \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \cos \frac{2n\pi}{3}.$$

Hint: decompose the sums in two or three summands respectively.

Problem 3.2.9

i) Prove that the series $\sum_{n=0}^{\infty} b_n 10^{-n}$, where $b_n \in \{0, 1, \ldots, 9\}$ for $n \geq 1$ and $b_0 \in \mathbb{Z}$, converges. What represents this series and why is it important?

ii) Compute the previous sum in the cases:

$$a) \quad b_n = 9, \quad n \geq 0; \qquad b) \quad b_n = \begin{cases} 1 & n = 2k \\ 2 & n = 2k+1 \end{cases}, \quad k \geq 0.$$

Problem 3.2.10 Let $\{a_n\}$ be a sequence of positive terms verifying $\lim_{n\to\infty} (a_n - n) = L > 0$.

i) Prove that the series $\sum_{n=1}^{\infty} n^{\alpha} \log(a_n/n)$ converges if and only if $\alpha < 0$.

ii) Compute the limits

$$P = \lim_{n\to\infty} \Big[\prod_{i=1}^{n} (\frac{a_i}{i}) \Big]^{1/n}, \qquad Q = \lim_{n\to\infty} \Big[\prod_{i=1}^{n} (\frac{a_i}{i})^i \Big]^{1/n}.$$

Hint: use problem 3.1.8.

Problem 3.2.11

i) Prove that the equation $\operatorname{tg} x = x$ has an unique solution λ_n on each interval $((2n-1)\pi/2, (2n+1)\pi/2)$, $n = 1, 2, 3, \cdots$

ii) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is convergent.

Problem 3.2.12 Study the convergence of the series

$$a) \quad \sum_{n=1}^{\infty} \sin(\pi n(1 + \frac{1}{2n^2})), \qquad b) \quad \sum_{n=1}^{\infty} \sin^2(\pi\sqrt{n^2+1}).$$

Hint: use problem 3.1.10.

Problem 3.2.13 Let the sequence defined as $x_{n+1} = \sqrt{1 + 2x_n} - 1$, $x_0 = 1$.

i) Prove that is convergent and compute its limit.

ii) Compute the limits

$$a) \quad \lim_{n\to\infty} \frac{x_{n+1}}{x_n}, \qquad b) \quad \lim_{n\to\infty} n x_n.$$

iii) Study the convergence of the series

$$a) \quad \sum_{n=0}^{\infty} x_n, \qquad b) \quad \sum_{n=0}^{\infty} x_n^2.$$

Hint: *ii.b*) apply Stolz conveniently; *iii*) apply part *ii*).

3.3 Taylor series

Problem 3.3.1 Find the interval of convergence of the following power series:

$$i)\ \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{n! x^n}{n^n}, \qquad iii)\ \sum_{n=1}^{\infty} \frac{x^n}{n 10^{n-1}},$$

$$iv)\ \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}, \qquad v)\ \sum_{n=1}^{\infty} (3-2x)^n, \qquad vi)\ \sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{2n}}.$$

Problem 3.3.2 Expand in power series the function $f_k(x) = \dfrac{1}{(1-x)^k}$, for $k = 1, 2, 3$.

Problem 3.3.3 Compute the radius of convergence and the sum of the following power series:

$$i)\ \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad ii)\ \sum_{n=0}^{\infty} (n+1) 2^{-n} x^n.$$

Problem 3.3.4 Expand in power series, showing the domain where the series is valid, of the functions:

$$i)\quad f(x) = \sin^2 x, \qquad ii)\quad f(x) = \frac{x}{a+bx}, \quad \text{with } a, b > 0,$$

$$iii)\quad f(x) = \log \sqrt{\frac{1+x}{1-x}}, \qquad iv)\quad f(x) = \frac{1}{2-x^2}, \qquad v)\ f(x) = \mathrm{e}^{x^2}.$$

Problem 3.3.5 Compute the sum of the following series

$$i)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{n}{2^n},$$

$$iii)\ \sum_{n=1}^{\infty} \frac{1}{n 2^n}, \qquad iv)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}.$$

Problem 3.3.6 Let C_0 be a circle of radius r.

i) Obtain a rectangle Q_0, inscribed in C_0, of maximum area.

ii) Let now C_1 be the maximum interior circle to that rectangle, concentric with C_0, and inscribe a rectangle Q_1 of maximum area in C_1. Compute the sum of the areas of the sequence $\{C_n\}_{n=0}^{\infty}$ of circles obtained by iterating the process.

Problem 3.3.7 Given the function $f(x) = \sum_{n=1}^{\infty} \dfrac{n^x}{n!}$, compute the values of $f(0)$, $f(1)$ and $f(2)$.

Problem 3.3.8 Find a function $f(x)$, with power series expansion, verifying

$$f'(x) = f(x) + x, \qquad f(0) = 2.$$

4 Integration in one variable

4.1 Antiderivatives

Problem 4.1.1 Find the following antiderivatives:

1. $\int x \operatorname{tg}^2(2x)\, dx$,
2. $\int \operatorname{tg}^3 x \sec^4 x\, dx$,
3. $\int \frac{\sqrt{x}+1}{x+3}\, dx$,
4. $\int \frac{(x+3)^3}{\sqrt{1-(x+1)^2}}\, dx$,
5. $\int \frac{x^2}{(x-1)^3}\, dx$,
6. $\int \frac{x^2+1}{\sqrt{x^2-1}}\, dx$,
7. $\int \frac{\sin^2 x \cos^5 x}{\operatorname{tg}^3 x}\, dx$,
8. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$,
9. $\int \mathrm{e}^x \sin \pi x\, dx$,
10. $\int \frac{dx}{\cos^4 x}$,
11. $\int \sin^2 x\, dx$,
12. $\int \sin^4 x\, dx$,
13. $\int \cos^2 x\, dx$,
14. $\int \cos^6 x\, dx$,
15. $\int \sin^2 x \cos^2 x\, dx$,
16. $\int \frac{dx}{3+\sqrt{2x+5}}$,
17. $\int \sqrt{\frac{x-1}{x+1}}\, dx$,
18. $\int \operatorname{arctg} \sqrt[3]{x}\, dx$,
19. $\int \sqrt{\sqrt{x}+1}\, dx$,
20. $\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\, dx$,
21. $\int \sqrt{2+\mathrm{e}^x}\, dx$,
22. $\int \mathrm{e}^{\sin x} \cos^3 x\, dx$,
23. $\int \sin^5 x\, dx$,
24. $\int \cos^3 x \sin^2 x\, dx$,
25. $\int \operatorname{tg}^2 x\, dx$,
26. $\int \operatorname{tg}^3 x\, dx$,
27. $\int x^3 \sqrt{1-x^2}\, dx$,
28. $\int \frac{\sin x + 3\cos x}{\sin x \cos x + 2 \sin x}\, dx$,
29. $\int \frac{\sin x + 3\cos x}{\sin x + 2\cos x}\, dx$,
30. $\int \operatorname{tg}^2(3x) \sec^3(3x)\, dx$,
31. $\int \frac{4x^4 - x^3 - 46x^2 - 20x + 153}{x^3 - 2x^2 - 9x + 18}\, dx$,
32. $\int \cos(\log x)\, dx$,
33. $\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x} + \mathrm{e}^x + 2}\, dx$,
34. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\, dx$,
35. $\int \frac{x^2}{(x^2+1)^{5/3}}\, dx$,
36. $\int \frac{2}{x^2-2x+2}\, dx$,
37. $\int \frac{dx}{\cos^2 x}$,
38. $\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$,
39. $\int \frac{x}{(x^2+1)^{5/2}}\, dx$,
40. $\int x^2(1-x^2)^{-3/2}\, dx$,
41. $\int \sqrt{\mathrm{e}^x - 1}\, dx$,
42. $\int \frac{2x^2+3}{x^2(x-1)}\, dx$,

43. $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$, 44. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$, 45. $\displaystyle\int x^2\sqrt{x-1}\,dx$,

46. $\displaystyle\int \sec^6 x\,dx$, 47. $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$, 48. $\displaystyle\int \frac{dx}{e^x-4e^{-x}}$,

49. $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$, 50. $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$, 51. $\displaystyle\int e^x\cos 2x\,dx$,

52. $\displaystyle\int x^2\log x\,dx$, 53. $\displaystyle\int \sin^3 x\cos^2 x\,dx$, 54. $\displaystyle\int \cos^4 x\,dx$,

55. $\displaystyle\int \mathrm{tg}^4 x\,dx$, 56. $\displaystyle\int \sec^3 x\,dx$, 57. $\displaystyle\int \frac{dx}{1-\sin x}$,

58. $\displaystyle\int \sin(\log x)dx$, 59. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$, 60. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$,

61. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$, 62. $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$, 63. $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$,

64. $\displaystyle\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}}$, 65. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$, 66. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$,

67. $\displaystyle\int x^m\log x\,dx$, 68. $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$, 69. $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$,

70. $\displaystyle\int \cos^2(\log x)\,dx$, 71. $\displaystyle\int (\log x)^3\,dx$, 72. $\displaystyle\int x(\log x)^2\,dx$.

Hint: IBP means integration by parts and CV change of variables.

1. IBP $dv = \mathrm{tg}^2(2x)dx$.
2. CV $t = \mathrm{tg}\ x$.
3. CV $t = \sqrt{x}$.
4. CV $t = \sqrt{1-(x+1)^2}$.
5. Partial fraction decomposition.
6. CV $x = \sec t$.
7. CV $t = \cos x$.
8. The derivative of the denominator almost appears in the numerator.
9. IBP twice using $dv = e^x dx$.
10. CV $t = \mathrm{tg}\, x$.
11, 12, 13, 14 and 15. Use the double angle formulas.
16. CV $t = \sqrt{2x+5}$.
17. CV $t = \sqrt{(x-1)/(x+1)}$.
18. CV $x = t^3$, after do IBP with $dv = t^2 dt$.
19. CV $t = \sqrt{\sqrt{x}+1}$.
20. CV $t = \sqrt{x+2}$.
21. CV $t = \sqrt{e^x+2}$.
22. CV $t = \sin x$, after do IBP twice with $dv = e^t dt$.
23. CV $t = \cos x$.
24. CV $t = \sin x$.
25. $\mathrm{tg}^2 x = \sec^2 x - 1$.
26. CV $t = \mathrm{tg}\, x$.
27. CV $t = \sqrt{1-x^2}$.
28 and 29. CV $t = \mathrm{tg}(x/2)$.
30. CV $t = \sin(3x)$.
31. Partial fraction decomposition.
32. IBP twice using $dv = dx$.
33. CV $t = e^x$.
34. CV $t = \sqrt{1+x^{1/3}}$.
35. CV $x = \mathrm{tg}\ t$.
36. Complete the square.

37. It is immediate.
38. CV $x+2=t^3$.
39. CV $t=(x^2+1)^{1/2}$.
40. CV $x=\sin t$.
41. CV $t=\sqrt{e^x-1}$.
42. Partial fraction decomposition.
43. CV $t=\sqrt{1-\sqrt{x}}$.
44. Multiply and divide by $1-\cos x$.
45. CV $t=\sqrt{x-1}$.
46. CV $t=\operatorname{tg} x$.
47. CV $t=1+x^2$.
48. CV $t=e^x$.
49. CV $t^2=1+x$.
50. CV $t^3=1-x$.
51. IBP twice using $dv=e^x dx$.
52. IBP $dv=x^2dx$.
53. CV $t=\cos x$.
54. Use the double angle formulas.
55. CV $t=\operatorname{tg} x$.
56. CV $t=\sin x$.
57. Multiply and divide by $1+\sin x$.
58. CV $t=\log x$.
59. CV $t=\sin x$.
60. CV $t^2=1+x^2$.
61. CV $t^2=e^{2x}-1$.
62. CV $t=e^x$.
63. Partial fraction decomposition.
64. CV $t^2=1-2x$.
65. CV $x=3\sin t$.
66. Complete the square.
67. IBP $dv=x^3x$.
68. CV $t=\sin x$.
69. CV $t^2=x^3$.
70. CV $t=\log x$ and use the double angle formulas.
71. IBP $dv=dx$.
72. IBP $dv=xdx$.

Problem 4.1.2 Find a continuous function f such that $f(0)=0$ and

$$f'(x)=\begin{cases}\dfrac{4-x^2}{(4+x^2)^2} & x<0\\ e^{\sqrt{x}} & x>0.\end{cases}$$

Problem 4.1.3 Compute $\int_a^b x\,dx$ using upper and lower sums associated to regular partitions of the interval $[a,b]$.

Problem 4.1.4

i) Prove that, if g is an odd and integrable function on $[-a,a]$, then $\int_{-a}^{a} g=0$. Apply the result to compute

$$\int_6^{10}\sin[\sin\{(x-8)^3\}]dx\,.$$

ii) Prove that, if h is an even and integrable function on $[-a,a]$, then $\int_{-a}^{a} h=2\int_0^a h$.

Problem 4.1.5 Prove and interpret the following identities:

$$i)\qquad \int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx,$$

$$ii)\qquad \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx,$$

$$iii)\qquad \int_{-a}^{a} [f(x) - f(-x)]\,dx = 0,$$

$$iv)\qquad \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx,$$

$$v)\qquad \int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}.$$

Problem 4.1.6 Let f be a periodic function of period T, integrable on $[0,T]$. Prove that:

i) for all integer n, we have

$$\int_a^b f = \int_{a+nT}^{b+nT} f;$$

ii) for all $a \in [0,T)$, we have

$$\int_a^{a+T} f = \int_0^T f;$$

Problem 4.1.7 Evaluate the following limits associating them to some definite integral:

$$i)\qquad \lim_{n\to\infty} \left[\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right],$$

$$ii)\qquad \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right],$$

$$iii)\qquad \lim_{n\to\infty} \frac{\sqrt[n]{e^2} + \sqrt[n]{e^4} + \cdots + \sqrt[n]{e^{2n}}}{n},$$

$$iv)\qquad \lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right].$$

Problem 4.1.8 Compute the limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n}\right)^{1/n}.$$

Problem 4.1.9 Evaluate $F(x) = \int_{-1}^{x} f(t)\,dt$ with $x \in [-1, 1]$, for the following functions:

i) $f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1; \end{cases}$ *ii)* $f(x) = |x|\, e^{-|x|}$;

iii) $f(x) = |x - 1/2|$; *iv)* $f(x) = \begin{cases} x^2 & -1 \le x < 0 \\ x^2 - 1 & 0 \le x \le 1; \end{cases}$

v) $f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ x + 1 & 0 < x \le 1; \end{cases}$ *vi)* $f(x) = \begin{cases} x + 2 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \\ -x + 2 & 1 \le x \le 2; \end{cases}$

vii) $f(x) = \max\{\sin(\pi x/2), \cos(\pi x/2)\}$.

Problem 4.1.10 Compute the following definite integrals, changing the limits of integration when making a change of variables:

$$i) \quad \int_0^{\log 2} \sqrt{\mathrm{e}^x - 1}\, dx, \qquad ii) \quad \int_1^2 \frac{\sqrt{x^2 - 1}}{x}\, dx.$$

4.2 The Fundamental Theorem of Calculus

Problem 4.2.1 Let $F(x) = \int_a^x f(t)\,dt$ with f integrable.

i) Prove that if $|f| \le M$ then $|F(x) - F(y)| \le M|x - y|$, implying the continuity of F.

ii) Is F differentiable necessarily? Under what conditions can we say that is differentiable?

Problem 4.2.2 Differentiate the following functions:

$$i) \quad F(x) = \int_{x^2}^{x^3} \frac{\mathrm{e}^t}{t}\, dt, \qquad ii) \quad F(x) = \int_{-x^3}^{x^3} \frac{dt}{1 + \sin^2 t},$$

$$iii) \quad F(x) = \int_3^{\int_1^x \sin^3 t dt} \frac{dt}{1 + \sin^6 t + t^2}, \qquad iv) \quad F(x) = \int_2^{\mathrm{e}^{\int_1^{x^2} \mathrm{tg}\sqrt{t}\, dt}} \frac{ds}{\log s}$$

$$v) \quad F(x) = \int_0^x x^2 f(t) dt, \quad \text{where } f \text{ is continuous on } I\!R,$$

$$vi) \quad F(x) = \sin\left(\int_0^x \sin\left(\int_0^y \sin^3 t dt\right) dy\right).$$

Problem 4.2.3 Compute the maximum and the minimum on $[1, \infty)$ of the function:

$$f(x) = \int_0^{x-1} \left(\mathrm{e}^{-t^2} - \mathrm{e}^{-2t}\right)\, dt.$$

Problem 4.2.4

i) Show that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution on $\mathbb{R}$ and that such solution lies on the interval $(0,1)$.

ii) Find and classify the local maxima and minima on $(0,\infty)$ of the function

$$G(x) = \int_0^{x^2} \sin t\, e^{\sin t}\,dt\,.$$

Problem 4.2.5 Find the tangent line to the graph of $y = \displaystyle\int_{x^2}^{\sqrt{\pi}/2} \operatorname{tg}(t^2)\,dt$ at $x = \sqrt[4]{\pi/4}$.

Problem 4.2.6 Calculate the following limits:

$$i)\quad \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{t^2}\,dt - x}{x^3}, \qquad ii)\quad \lim_{x\to 0} \frac{\cos x \displaystyle\int_0^x \sin t^3\,dt}{x^4}.$$

Problem 4.2.7 Compute one-sided limits at the origin of the function

$$f(x) = \frac{x - \sin x + \displaystyle\int_0^{x^2} \operatorname{tg}(\sqrt{t})\,dt}{2x^3}\,.$$

Problem 4.2.8 Consider the function $f(x) = \displaystyle\int_0^{x^2} \frac{\sin t}{t}\,dt$.

i) Using the Taylor series of the sine function, find the Taylor series of f around the origin.

ii) Compute $\displaystyle\lim_{x\to 0} \frac{f(x)}{1-\cos x}$.

iii) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} f(1/n)$.

Problem 4.2.9 If the integral $\displaystyle\int_{-1/x}^{x} \frac{dt}{a^2+t^2}$ does not depend on x, without computing the integral, find a.

Problem 4.2.10 Consider the functions $f(x) = e^{x^2} - x^2 - 1$, $g(x) = 3 + \displaystyle\int_0^x f(t)\,dt$.

i) Write down the Taylor polynomial of g around the origin.

ii) Determine if g has a maximum, a minimum or an inflection point around the origin.

Problem 4.2.11 Let g be a derivable function verifying the equation

$$t = \int_0^{(g(t))^2} \frac{\sin x}{x}\,dx.$$

i) Write down $g'(t)$ in terms of $g(t)$.

ii) Compute $(g^{-1})'(x)$.

Problem 4.2.12 The equation

$$\int_0^{g(x)} (e^{t^2} + e^{-t^2})\,dt - x^3 - 3\,\mathrm{arctg}\,x = 0,$$

defines a differentiable and one to one function g differentiable on $\mathbb{R}$. Compute:

i) $g(0)$, $g'(0)$ and $(g^{-1})'(0)$;

ii) $\displaystyle\lim_{x\to 0} \frac{g^{-1}(x)}{g(x)}$.

Problem 4.2.13 Find the explicit formula of a continuous function, $f : \mathbb{R} \to \mathbb{R}$, verifying

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C.$$

Next, find the value of C.

4.3 Applications

Problem 4.3.1 Find the area enclosed by the following curves:

i) $y = x^2$, $y = (x-2)^2$, $y = (2-x)/6$;

ii) $x^2 + y^2 = 1$, $x^2 + y^2 = 2x$;

iii) $y = \dfrac{1-x}{1+x}$, $y = \dfrac{2-x}{1+x}$, $y = 0$, $y = 1$;

iv) loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Problem 4.3.2 Find the area bounded by the graph of $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the horizontal axis.

Problem 4.3.3 Find the area enclosed by the following curves given in parametric and polar coordinates:

i) loop: $x = t^2 + 1$, $y = t(t^2 - 4)$, $\quad -2 \le t \le 2$;

ii) cycloid: $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $\quad 0 \le t \le 2\pi$, and axis X;

iii) spiral of Archimedes: $r = a\theta$, $\quad 0 \le \theta \le 2\pi$ and the line segment $\{0 \le x \le 2\pi a,\ y = 0\}$;

iv) one leaf of the three-leaved rose: $r = a\cos 3\theta$, $\quad -\pi/6 \le \theta \le \pi/6$;

v) half of the lemniscate: $r = a\sqrt{\cos 2\theta}$, $\quad -\pi/4 \le \theta \le \pi/4$.

Problem 4.3.4

i) Find the area between the graph of the function $f(x) = \dfrac{x^2-4}{x^2+4}$ and its asymptote.

ii) Find the area enclosed by the graph of the function $f(x) = \dfrac{1}{1+e^{-x}}$, its asymptote at $x \to +\infty$ and the vertical axis.

iii) Compute the area enclosed by the graph of the function $f(x) = \dfrac{1-x}{(x+1)^2\sqrt{x}}$ and its asymptotes.

iv) Find the area enclosed by the graphs of the functions $f_1(x) = \dfrac{x-4}{(x+4)\sqrt{x}}$ and $f_2(x) = \dfrac{1}{\sqrt{x}}$ for $x \geq 4$.

Problem 4.3.5 Let A be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Compute the area of A and the revolution volume obtained by rotating A about the horizontal axis.

Problem 4.3.6 Evaluate the volumes formed by revolving the following regions about the X axis:

i) $0 \leq y \leq 1 + \sin x, \quad 0 \leq x \leq 2\pi$;

ii) $x^2 + (y - 2a)^2 \leq a^2$ (the graph is a torus);

iii) $R^2 \leq x^2 + y^2 \leq 4R^2$ (an spherical ring);

iv) the surface bounded by the curves $y = \sin x$ and $y = x$, with $x \in [0, \pi]$;

v) $x = t - \sin t,\ 0 \leq y \leq 1 - \cos t, \quad 0 \leq t \leq 2\pi$.

Problem 4.3.7

i) Compute the volumes formed by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$ about the X and Y axes.

ii) Compute the volume of the solid with base the previous ellipse and whose perpendicular sections to the OX axis are isosceles triangles of height 2.

Problem 4.3.8

i) Compute the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$.

ii) Compute the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$.

iii) Show that the previous problem (part *i*)) is a particular instance of this one.

Hint: observe that when cutting the ellipsoid in parallel sections to the coordinate planes, we obtain ellipses.

Problem 4.3.9 Find the length of the following graphs:

i) catenary: $y = e^{x/2} + e^{-x/2}, \quad 0 \leq x \leq 2$;

ii) cycloid: $x(t) = a(t - \sin t),\ y(t) = a(1 - \cos t), \quad 0 \leq t \leq 2\pi$;

iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4$;

iv) tractrix: $y = a \log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}, \quad a/2 \leq x \leq a$;

v) cardioid: $r = 1 + \cos\theta, \quad 0 \leq \theta \leq 2\pi$;

vi) circular helix: $x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) = bt, \quad 0 \leq t \leq 2\pi$.