

# Universidad Carlos III de Madrid

## Escuela Politécnica Superior

DEPARTAMENTO DE MATEMÁTICAS

First Course

Bachelor's Degree in Telematics, Communication System and Audiovisual System Engineering

Calculus I. Final exam, January 22nd, 2009

Time length: 3:30 hours

**Problem 1. (3 p.)** Consider the function defined as:

$$f(x) = \begin{cases} \ln(x^2) & x \in (-\infty, -1) \cup (1, \infty) \\ -x^2 + 1 & x \in [-1, 1] \end{cases}$$

- Analyze its continuity and differentiability.
- Compute its global and local maxima and minima.
- Sketch its graph.

**Problem 2. (1 p.)** Let  $\{a_n\}$  be a sequence of positive terms verifying  $\lim_{n \rightarrow \infty} a_n = L$ , compute the limit

$$\lim_{n \rightarrow \infty} \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}.$$

**Problem 3. (2 p.)**

- Study if the following series converges or not

$$\sum_{n=1}^{\infty} \frac{e^{2n} \sqrt{n}}{(n!)^2}.$$

- Sum the series

$$\sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{e^n} - \frac{\sqrt{n+1}}{e^{n+1}} \right).$$

**Problem 4. (2 p.)** Let  $f(x)$  be the function defined as

$$f(x) = \int_0^{x^2} \frac{\log(1+t)}{t} dt$$

- Compute the Taylor series of  $f(x)$  around  $x = 0$  using the Taylor series of  $\log(1+t)$ .
- Compute the Taylor series of  $f'(x)$  around  $x = 0$ . Analyze whether  $x = 0$  is a local maximum or minimum of  $f(x)$ .

**Problem 5. (2 p.)** Find the following antiderivatives

$$\int \frac{dx}{x^4 + 2x^2}, \quad \int_1^{e^2} \log(\sqrt{x}) dx.$$

1. a) Continuity: As  $\ln(x^2)$  and  $-x^2 + 1$  are continuous on their domains, we have to check the continuity only on  $\{-1, 1\}$ :

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) = 0$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^-} f(x) = f(-1) = 0$$

Therefore,  $f(x)$  is continuous on  $\mathbb{R}$ .

Differentiability:

$$f'(x) = \begin{cases} 2/x & x \in (-\infty, -1) \cup (1, \infty) \\ -2x & x \in (-1, 1) \end{cases}$$

$$\lim_{x \rightarrow 1^+} f'(x) = 2 \neq \lim_{x \rightarrow 1^-} f'(x) = -2$$

$$\lim_{x \rightarrow -1^+} f'(x) = 2 \neq \lim_{x \rightarrow -1^-} f'(x) = -2$$

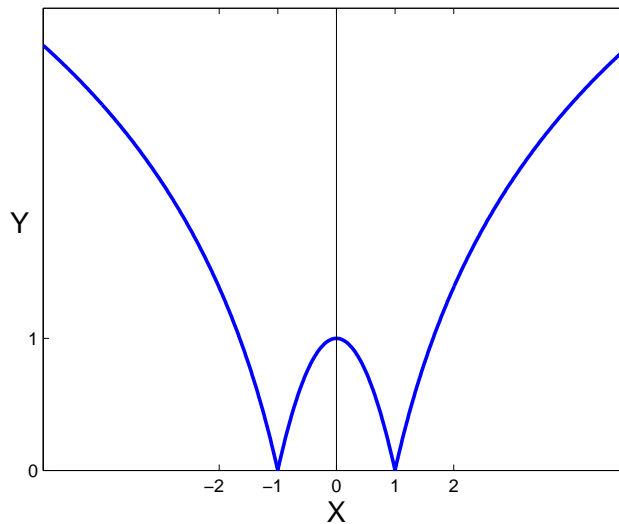
Therefore,  $f(x)$  is differentiable on  $\mathbb{R} - \{-1, 1\}$ .

- b) The critical points of  $f$  are the points in the domain where the function is not differentiable, that is,  $\{-1, 1\}$ , and the points whose derivative vanishes,  $f'(x) = 0 \Rightarrow x = 0$ . So we have to study the points  $\{-1, 0, 1\}$ . Using the first derivative test, we obtain

$x$	$f'(x)$	
-1	$- \rightarrow +$	local minimum
0	$+ \rightarrow -$	local maximum
1	$- \rightarrow +$	local minimum

Since  $f(-1) = f(1) = 0$  and  $f(x) \geq 0$ , these points are global minima. As  $\lim_{x \rightarrow \infty} f(x) = \infty$  there is no global maximum.

- c) The graph of  $f$  is



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2.  $\lim_{n \rightarrow \infty} \frac{a_1^2 + a_2^2 + \dots + a_n^2}{n} = \lim_{Stolz \ n \rightarrow \infty} \frac{a_1^2 + a_2^2 + \dots + a_n^2 - (a_1^2 + a_2^2 + \dots + a_{n-1}^2)}{n - (n-1)} = \lim_{n \rightarrow \infty} a_n^2 = L^2.$

3. a) By the quotient test, it converges:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{e^{2n+2} \sqrt{n+1}}{((n+1)!)^2} \frac{(n!)^2}{e^{2n} \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{e^2}{(n+1)^2} \frac{\sqrt{n+1}}{\sqrt{n}} = 0 < 1.$$

b) It is a telescoping series, therefore,

$$\sum_{n=1}^{\infty} \left( \frac{\sqrt{n}}{e^n} - \frac{\sqrt{n+1}}{e^{n+1}} \right) = \frac{1}{e} - \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} = \frac{1}{e}.$$

4. a)

$$\begin{aligned} f(x) &= \int_0^{x^2} \frac{\log(1+t)}{t} dt = \int_0^{x^2} \frac{t - t^2/2 + \dots + (-1)^{n+1} t^n/n + \dots}{t} dt = \\ &= \int_0^{x^2} (1 - t/2 + \dots + (-1)^{n+1} t^{n-1}/n + \dots) dt = \\ &= \left[ t - \frac{t^2}{2 \cdot 2} + \frac{t^3}{3 \cdot 3} + \dots + (-1)^{n+1} \frac{t^n}{n^2} + \dots \right]_0^{x^2} = \\ &= x^2 - \frac{x^4}{4} + \frac{x^6}{9} + \dots + (-1)^{n+1} \frac{x^{2n}}{n^2} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n^2}. \end{aligned}$$

b) 
$$f'(x) = 2x - x^3 + \frac{6}{9}x^5 + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} x^{2n-1}.$$

$f(0)$  is a local minimum.

5. In the first integral we must do partial fraction decomposition:

$$\frac{1}{x^2(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2} \rightarrow A = C = 0, B = 1/2, D = -1/2 \rightarrow$$

$$\int \frac{dx}{x^4+2x^2} = \int \frac{dx}{x^2(x^2+2)} = \frac{1}{2} \int \frac{dx}{x^2} - \frac{1}{2} \int \frac{dx}{x^2+2} = \frac{1}{2x} - \frac{1}{2\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + c.$$

In the second integral we change variables first,  $\sqrt{x} = t$ , and then we integrate by parts using  $u = \log t$  and  $dv = 2t dt$ , hence the integral is

$$\int_1^{e^2} \log \sqrt{x} dx = \int_1^e \log t \cdot 2t dt = \log t \cdot t^2 \Big|_1^e - \int_1^e t dt = e^2 - \frac{e^2}{2} + \frac{1}{2} = \frac{e^2}{2} + \frac{1}{2}.$$