

Universidad Carlos III de Madrid

Escuela Politécnica Superior

DEPARTAMENTO DE MATEMÁTICAS

First Course. Bachelor's Degree in

Telematics, Communication System and Audiovisual System Engineering

GRADE

Calculus I. Second Test, November 25, 2008

Surname..... Name.....
D.N.I..... Group.....

Time length: 80 min.

1. (Class problem) Approximate e using a Taylor Polynomial, with error less than 0.001.
How many terms must we use in the polynomial?

[2 p.]

2. (Problem 3.1.4) Compute the limit

$$\lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\log n}$$

[2 p.]

3. (Problems 3.2.1, 3.2.4) Analyze the convergence of the following series

1. $\sum_{n=2}^{\infty} \log\left(\frac{n+1}{n}\right)$

2. $\sum_{n=1}^{\infty} (-1)^n \left(\arctan \frac{1}{n}\right)^2$

Hint: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)}$.

[4 p.]

4. (Problem 3.3.4) Compute the Taylor Series, the radius and the interval of convergence of

$$f(x) = \frac{1}{2 - x^2}$$

Hint: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n \dots$

[2 p.]

ANSWERS:

$$1. e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + E, \quad E(x) = \frac{e^z}{(n+1)!} x^{n+1}, \quad z \in (0, x),$$

$$E(1) = \frac{e^z}{(n+1)!} < \frac{3}{(n+1)!} < 0.001 \Rightarrow 3000 < (n+1)! \Rightarrow n = 6$$

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = 2.718.$$

$$2. \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\log n} \stackrel{\text{Stolz}}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\log\left(\frac{n+1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\log(1+1/n)} = 1.$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{\log(1+1/x)} \stackrel{\text{Taylor}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{1/x} = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1.$$

3.

1. By the limit comparison test: $\lim_{n \rightarrow \infty} \frac{\log\left(\frac{n+1}{n}\right)}{1/n} = \lim_{n \rightarrow \infty} \log(1+1/n)^n = \log e = 1 \Rightarrow$
the series diverges.

2. As $(\arctan 1/n)^2 = \left(\frac{1}{n} + o\left(\frac{1}{n^2}\right)\right)^2 = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$, by the limit comparison test, the series is absolutely convergent:

$$\lim_{n \rightarrow \infty} \frac{(\arctan n)^2}{1/n^2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)}{1/n^2} = 1.$$

$$4. f(x) = \frac{1}{2-x^2} = \frac{1}{2} \cdot \frac{1}{1-x^2/2} = \frac{1}{2} \left(1 + \frac{x^2}{2} + \left(\frac{x^2}{2}\right)^2 + \left(\frac{x^2}{2}\right)^3 + \dots\right) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n},$$

hence $a_n = \frac{1}{2^{n/2}} \Rightarrow \frac{1}{\rho} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{2^{n/2}}} = 1/\sqrt{2}$ so the radius of convergence is $\rho = \sqrt{2}$.

To study the interval of convergence, we check the end points, as $f(\pm\sqrt{2}) = 1 + 1 + 1 + 1 + \dots$ is divergent, we conclude that the interval of convergence is $(-\sqrt{2}, \sqrt{2})$.
