2 Differential Calculus in one variable

2.1 Derivatives

The derivative provides a way to calculate the rate of change of a function. We compute the average velocity as the rate between the distance during a time interval h and the length of the time interval

$$v_{average} = \frac{x(t+h) - x(t)}{h}$$

this is just the slope m of the line passing through the points $(t, x(t))$ and $(t+h, x(t+h))$. If we want to compute the velocity at a time t we should take the limit

$$v(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \frac{d}{dt} x(t)$$

Definition 2.1.1 *A function f is* **differentiable** *at x* $\iff$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is a finite number.

If f is differentiable then $f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called the **derivative** *of f at x.*

Note. The function f' exist for the points on the domain of f such that the limit exists and is finite.

Definition 2.1.2 (Alternative def.)

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Example 2.1.1 *Differentiate the functions* x^3, $\sqrt{x}$, $\sin x$, e^x *and* $|x|$.

Tangent Line The line passing through $(x_0, f(x_0))$ with slope $m = f'(x_0)$ is the tangent line to $f(x)$ at x_0: $y = m(x - x_0) + f(x_0)$.

Example 2.1.2 *Find the tangent line of* $\sin x$ *at* $x_0 = 0$.

Properties

1. $(c_1 f + c_2 g)' = c_1 f' + c_2 g', \quad c_1, c_2 \in \mathbb{R}$.
2. $(f \cdot g)' = f'g + fg'$
3. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Theorem 2.1.1 (The Chain Rule) *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x, and verifies*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Example 2.1.3 *Prove that*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Use that identity to compute $(\arctan x)'$ *and* $(\ln x)'$.

Basic derivatives

1. $c' = 0$
2. $(x^n)' = nx^{n-1}$
3. $(e^x)' = e^x, \qquad (\log x)' = \dfrac{1}{x}$
4. $(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = \dfrac{1}{\cos^2 x}$
5. $(\arctan x)' = \dfrac{1}{1+x^2}, \qquad (\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$
6. $(\sinh x)' = \cosh, \qquad (\cosh x)' = \sinh x$

Example 2.1.4 *Differentiate* $\tan x$ *and* $\sinh x$.

Theorem 2.1.2 f *differentiable* $\Rightarrow$ f *continuous*

Theorem 2.1.3 (Rolle's Theorem) *Let* f *be differentiable on* (a, b) *and continuous on* $[a, b]$. *If* $f(a) = f(b)$, *then there is at least a number* $c \in (a, b)$ *such that*

$$f'(c) = 0.$$

Theorem 2.1.4 (Mean Value Theorem) *Let* f *be differentiable on* (a, b) *and continuous on* $[a, b]$, *then there is at least a number* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently $f(b) - f(a) = f'(c)(b - a)$.

Theorem 2.1.5 (L'Hôpital's Rule) *Let* f *and* g *be differentiable functions on* (a, b), *except possibly at the point* $x_0 \in (a, b)$. *If* $\lim_{x \to x_0} \dfrac{f(x)}{g(x)}$ *is the indeterminate form* $\dfrac{0}{0}$, *then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},$$

whenever $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ *exists or it is infinite.*

Extensions L'Hôpital's Rule can be applied also in the following cases:

- If the indeterminate form is $\frac{\infty}{\infty}$ with all the possible signs.
- If the limit is taken when $x_0 \to \pm\infty$
- To one-sided limits

Implicit differentiation $F(x, y) = 0$ differentiate with respect to x and then obtain $\dfrac{dy}{dx}$.

Example 2.1.5 *Compute* y' *from* $F(x, y) = x\exp(y) - y + 1 = 0$, *and find the tangent line at* $(-1, 0)$.

Higher order derivatives We can compute the derivative of a derivative:

$$\frac{d^2 f}{dx^2} = f''(x), \qquad \frac{d^3 f}{dx^3} = f'''(x), \cdots, \frac{d^n f}{dx^n} = f^{(n)}(x)$$

Example 2.1.6 *Let* $f(x) = x\exp x$, *compute* $f'(x),\ f''(x), \cdots, f^{(n)}(x)$.

2.2 Extrema

Definition 2.2.1 *Let f be a function defined on an interval I:*

- *$f(x_m)$ is the* **global** *(or absolute)* **minimum** *of f on I if $f(x_m) \leq f(x),\ \forall x \in I$.*
- *$f(x_M)$ is the* **global** *(or absolute)* **maximum** *of f on I if $f(x_M) \geq f(x),\ \forall x \in I$.*

Note. Remember that if f is continuous, on a bounded and closed interval $[a, b]$ the function always reaches its global maximum and minimum.

Definition 2.2.2 *Let f be a function defined on an interval I, if we have an open interval I_1 containing x_0*

- *$f(x_0)$ is a* **local** *(or relative)* **minimum** *of f on I if $f(x_0) \leq f(x),\ \forall x \in I_1$.*
- *$f(x_0)$ is a* **local** *(or relative)* **maximum** *f on I if $f(x_0) \geq f(x),\ \forall x \in I_1$.*

Example 2.2.1 $f(x) = x^2,\ f(x) = |x|$.

Definition 2.2.3 *Let f be a function defined at x_0. f has a* **critical point** *at x_0 if $f'(x_0) = 0$ or $f'(x_0)$ does not exist.*

Theorem 2.2.1 *If f has a local maximum or minimum at x_0, then x_0 is a critical point of f.*

Finding the global extrema of a function f on a closed interval $[a, b]$

1. Compute the critical points of f on (a, b): $f'(x_0) = 0$ or $f'(x_0)$ does not exist.
2. Evaluate f at each critical point of (a, b).
3. Evaluate f at the endpoints of the interval $f(a)$ and $f(b)$.
4. The smallest value is the global minimum and the greatest one, the global maximum.

Example 2.2.2 $f(x) = x^3,\ [-1, 1]$.

Definition 2.2.4

- *f is an* **increasing function** *on an interval I if $\forall x_1, x_2 \in I$ with $x_1 < x_2$ we have that $f(x_1) < f(x_2)$.*
- *f is a* **decreasing function** *on an interval I if $\forall x_1, x_2 \in I$ with $x_1 < x_2$ we have that $f(x_1) > f(x_2)$.*

Theorem 2.2.2 *Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b)*

1. *If $f'(x) > 0,\ \forall x \in (a, b)$ then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0,\ \forall x \in (a, b)$ then f is decreasing on $[a, b]$.*
3. *If $f'(x) = 0,\ \forall x \in (a, b)$ then f is constant on $[a, b]$.*

Test of the first derivative

	$x < x_0$	$x > x_0$	x_0 (critical point)
$f'(x)$	$-$	$+$	local minimum
	$+$	$-$	local maximum
	$-$	$-$	neither
	$+$	$+$	neither

Example 2.2.3 $f(x) = x^3 - 3x^2/2$.

Definition 2.2.5 *Let f be differentiable on an open interval I. The graph of f is*

- **convex** *(concave upwards) on I if f' is increasing.*
- **concave** *(concave downwards) on I if f' is decreasing.*

Theorem 2.2.3 *Let f be a function twice differentiable on an open interval I*

- *If $f''(x) > 0$, $\forall x \in I$, then the graph of f is convex on I.*
- *If $f''(x) < 0$, $\forall x \in I$, then the graph of f is concave on I.*

Definition 2.2.6 *Let f be a continuous function on an open interval I, and let $x_0 \in I$. f has an* **inflection point** *at x_0 if the concavity changes at x_0 (convex $\leftrightarrow$ concave).*

Theorem 2.2.4 *If x_0 is an inflection point of f, then $f''(x_0) = 0$ or $f''(x_0)$ does not exist.*

Theorem 2.2.5 *Let f be a function such that $f'(x_0) = 0$ and twice differentiable on an open interval containing x_0*

- *if $f''(x_0) > 0$, then f has a local minimum at x_0.*
- *if $f''(x_0) < 0$, then f has a local maximum at x_0.*

If $f''(x_0) = 0$ the test does not work, it can be anything.

$f'(x_0)$	$f''(x_0)$	graph
$+$	$-$	increasing, concave
$-$	$-$	decreasing, concave
$+$	$+$	increasing, convex
$-$	$+$	decreasing, convex
0	$+$	local minimum
0	$-$	local maximum
0	0	?

2.3 Graphs

1. Domain
2. Intersection with x-axis $\rightarrow f(x) = 0$
   Intersection with y-axis $\rightarrow f(0) = y$
3. Symmetries

   $f(-x) = +f(x) \rightarrow$ even
   $f(-x) = -f(x) \rightarrow$ odd

   Periodicity $\rightarrow f(x+T) = f(x)$
4. Asymptotes:

   Vertical $\rightarrow \lim_{x \to x_0} f(x) = \pm\infty$
   Horizontal $\rightarrow \lim_{x \to \pm\infty} f(x) = H$
   Oblique $\rightarrow \lim_{x \to \pm\infty} f(x) - (mx + b) = 0 \rightarrow m = \lim_{x \to \infty} \frac{f(x)}{x},\ b = \lim_{x \to \infty} (f(x) - mx)$
5. Continuity: $\lim_{x \to x_0} f(x) = f(x_0)$
6. Derivative: monotonicity and critical points

   $f'(x) > 0$ increasing
   $f'(x) < 0$ decreasing
   $f'(x) = 0$ or $f'(x)$ does not exist $\rightarrow$ **critical points**
7. Local maxima and minima: $x_0 \rightarrow$ critical point

$f'(x_0) = 0,\ f''(x_0) > 0$	local minimum
$f'(x_0) = 0,\ f''(x_0) < 0$	local maximum
$f'(x) : - \mapsto +$	local minimum
$f'(x) : + \mapsto -$	local maximum
8. Concavity

   $f''(x) > 0$ convex
   $f''(x) < 0$ concave
9. Inflection points. Concavity changes. $f''(x_0) = 0$ or $\nexists f''(x_0)$
10. Global maxima and minima

2.4 Taylor polynomial

The idea is to approximate a function $f(x)$ by a polynomial $P(x)$. The Taylor polynomial is the best polynomial that approximates a function at a point x_0.

If we approximate $f(x)$ by $\begin{cases} \text{a constant} & \rightarrow P(x) = f(x_0) \\ \text{a line} & \rightarrow P(x) = f(x_0) + f'(x_0)(x - x_0) \end{cases}$

Definition 2.4.1 *If f is differentiable n times at x_0, then the polynomial*

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is the Taylor polynomial of degree n of f at x_0.

Note. When $x_0 = 0$ the polynomial is called Mac Laurin Polynomial.
Error The polynomial approximates $f(x)$, so we have an **error**
$|R_n(x)| = |f(x) - P(x)|$. There are many formulas for the error, but the idea of all of them is that

$$\lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x - x_0)^n} = 0.$$

$\rightarrow R_n(x) = o\big((x - x_0)^n\big)$. Notation: $f(x) = o\big(g(x)\big)$ when $x \to x_0 \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

In the following theorem we give a formula for the error $|R_n(x)|$:

Theorem 2.4.1 *Let $f(x)$ be a function differentiable $n + 1$ times on an open interval I, then $\forall x_0, x \in I$ we have that*

$$f(x) = P_n(x) + R_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x),$$

$R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{(n+1)}$, ξ is a point in the open interval defined by x_0 and x.