1 Functions of a real variable

1.3 Limits

If $f(x)$ is a function defined for all x near x_0, but not necessarily at $x = x_0$ itself, if the value $f(x)$ of f approximates a number L as x gets closer to a number x_0, we say that *L is the limit of $f(x)$ as x approaches x_0.*

$$f(x) = \frac{2x^2 - 7x + 3}{x - 3}$$

x	2.9	2.9999	2.999999	$\to 3 \leftarrow$	3.000001	3.0001	3.1
$f(x)$	4.8	4.9998	4.999998	$\to 5 \leftarrow$	5.000002	5.0002	5.2

$$\lim_{x\to 3} f(x) = 5$$

$$f(x) = \frac{2x^2 - 7x + 3}{x - 3} = \frac{(2x-1)(x-3)}{x-3} \underset{x\neq 3}{=} 2x - 1.$$

Definition 1.3.1 (Weierstrass, $\epsilon - \delta$ definition) *Let f be a function defined on an open interval containing x_0 (except possibly at x_0) and let L be a real number, then*

$$\lim_{x\to x_0} f(x) = L,$$

if $\forall \epsilon\, 0,\ \exists \delta > 0,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$

Example 1.3.1 *Use the $\varepsilon - \delta$ to prove that* $\lim\limits_{x\to 3}(2x - 1) = 5.$

Note. A useful general rule is to write down $f(x) = L$ and then to express it in terms of $x - x_0$ as much as possible, by writing $x = (x - x_0) + x_0$.

Note. In the $\varepsilon - \delta$ def, L and x_0 are finite numbers. We have similar definitions for $x \to \pm\infty$ and also if the limit is not a finite number. We define also one-sided limits: from the right of x_0: $\lim\limits_{x\to x_0^+} f(x)$, from the left of x_0: $\lim\limits_{x\to x_0^-} f(x)$.

Indeterminate forms $\infty - \infty,\ \dfrac{\infty}{\infty},\ \infty \cdot 0,\ \dfrac{0}{0},\ \infty^0,\ 1^\infty,\ 0^0.$

Unless we arrive at an indetermination, we have the following properties.
Properties Assume that $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist, where x_0 is a finite number or $\pm\infty$

a) $\lim\limits_{x\to x_0} (c_1 f(x) + c_2 g(x)) = c_1 \lim\limits_{x\to x_0} f(x) + c_2 \lim\limits_{x\to x_0} g(x), \quad c_1, c_2 \in \mathbb{R}$

b) $\lim\limits_{x\to x_0} (f(x) g(x)) = \lim\limits_{x\to x_0} f(x) \lim\limits_{x\to x_0} g(x)$

c) $\lim\limits_{x\to x_0} \left(\dfrac{1}{f(x)}\right) = \dfrac{1}{\lim\limits_{x\to x_0} f(x)}$, if $\lim\limits_{x\to x_0} f(x) \neq 0$

d) Replacement rule: if f and g agree for all x near x_0 (not necessarily including x_0), then $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x)$

e) Composite function rule: if $\lim\limits_{x\to x_0} f(x) = L$ and $\lim\limits_{x\to L} h(x) = h(L)$, then $\lim\limits_{x\to x_0} h(f(x)) = h(\lim\limits_{x\to x_0} f(x))$

Basic limits. Let x_0 be a finite number

a) $\lim_{x\to x_0} c = c, \quad \lim_{x\to\pm\infty} c = c$

b) $\lim_{x\to x_0} x = x_0, \quad \lim_{x\to\pm\infty} x = \pm\infty, \quad \lim_{x\to\pm\infty} \frac{1}{x} = 0$

c) $\lim_{x\to x_0} x^n = x_0^n$, $n \in \mathbb{N}$

d) $\lim_{x\to x_0} \sqrt[n]{x} = \sqrt[n]{x_0}$, ($\forall x_0$ in its domain)

e) $\lim_{x\to x_0} f(x) = f(x_0) \to$ trigonometric functions on their domain

Example 1.3.2 *Prove that* $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

Lemma 1.3.1 (Squeeze or Sandwich Lemma) *Let I be an interval such that $x_0 \in I$. Let f, g and h be functions defined on I, except possibly at x_0 itself. If $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$ and, $\forall x \in I$, $x \neq x_0$, $h(x) \leq f(x) \leq g(x)$. Then*

$$\lim_{x\to x_0} f(x) = L.$$

Example 1.3.3 *Use the Sandwich Lemma to prove that* $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

1.4 Continuity

A continuous function is a function for which intuitively, small changes in the input result in small changes in the output.

Definition 1.4.1 *Let f be a function defined on $(x_0 - p, x_0 + p)$, $p > 0$*

$$\boxed{f \text{ is } \textbf{continuous} \text{ at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0).}$$

Notice that the function must be defined at x_0 to be continuous at that point. If the function is not continuous at x_0 we say that it is **discontinuous** at that point:

- $f(x)$ is <u>discontinuous</u> at x_0 if $\begin{cases} \lim_{x \to x_0} f(x) \text{ does not exist, or} \\ \text{the limit exist but is not equal to } f(x_0) \end{cases}$

Definition 1.4.2 *A function $f(x)$ is continuous on*

- $\mathbb{R}$*, if it is continuous at every point.*
- (a, b)*, if it is continuous at each point of the interval.*
- $[a, b]$*, if it is continuous on (a, b) and*

$$f(a) = \lim_{x \to a^+} f(x), \; f(b) = \lim_{x \to b^-} f(x).$$

Basic properties
Let $f, \; g$ be continuous at x_0, then the following functions are continuous at x_0

a) $c_1 f + c_2 g, \quad c_1, c_2 \in \mathbb{R}$

b) fg

c) $\dfrac{1}{f}$, if $f(x_0) \neq 0$

d) Composite function: if g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0

Some continuous functions
The following functions are continuous on their <u>domains</u>

a) $p(x)$ polynomials

b) $\dfrac{p(x)}{q(x)}$ rational functions

c) $\sqrt[n]{x}$

d) trigonometric functions: $\sin(x), \; \cos(x), \; \tan(x), \; \arcsin(x), \ldots$

e) hyperbolic functions $\sinh(x), \; \cosh(x), \ldots$

f) $\exp(x)$ and $\ln(x)$

Example 1.4.1

$$f(x) = \begin{cases} \frac{2x^2-7x+3}{x-3}, & x \neq 1,3 \\ 5 & x = 3 \\ -1 & x = 1 \end{cases}$$

Theorem 1.4.1 (Intermediate Value Theorem) *If $f(x)$ is a continuous function on $[a,b]$ and K is any number between $f(a)$ and $f(b)$, then there is a $c \in [a,b]$ such that $f(c) = K$.*

Theorem 1.4.2 (Bolzano's Theorem) *If $f(x)$ is a continuous function on $[a,b]$ and $f(a) \cdot f(b) < 0$, then there is a $c \in (a,b)$ such that $f(c) = 0$.*

Example 1.4.2 *Sketch the graph of*

$$f(x) = \frac{1}{x}$$

on *a)* $\mathbb{R}$ *b)* $[0,\infty)$ *c)* $[1,\infty)$ *d)* $[1,10]$

A function f is **bounded** if the set of its values is bounded. That is, if there exists a number $M > 0$ such that $|f(x)| \leq M$, for all x in its domain.

Theorem 1.4.3 (Extreme Value Theorem) *If $f(x)$ is a continuous function on the closed interval $[a,b]$, then f attains its maximum and minimum value. That is, there exist numbers x_m and x_M in $[a,b]$ such that:*

$$f(x_M) \geq f(x) \geq f(x_m) \quad \text{for all } x \in [a,b].$$